Lecture 22: Rules for Solving IVP

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1 Review

Recall that last time we discussed the numerical solution of first order differential equations. When solving differential equations, we are looking for a function $y(t)$, given an expression for the derivative $y'(t)$ in terms of t and $y(t)$:

$$y'(t) = f(t, y(t))$$

Last time we showed that this problem is ill-formed, since there are infinitely many solutions (in fact, for each point $(a, y(a))$, there is a unique solution $y(t)$). In order to make this problem well-formed, we must also provide an *initial value* for $y(t)$, giving us an *initial value problem*

PROBLEM 1.1 (Initial Value Problem). *Given*

$$\begin{aligned} y'(t) &= f(t, y(t)) \\ y(a) &= y_0 \end{aligned}$$

find $y(b)$

Last time, we discussed some simple methods for solving initial value problems. All of our rules follow the same general scheme

1. Partition the interval $[a, b]$ into N subintervals of equal length $h = (b - a)/N$ using points $(t_0, t_1, \ldots, t_N)$.

2. Denote values for the function $y_i = y(t_i)$ and derivative $y'_i = y'(t_i)$ at each point. We will compute approximations for these values $Y_i \approx y_i$ and $Y'_i \approx y'_i$.

3. Start with $Y_0 = y_0$, $Y'_0 = f(t_0, Y_0)$, then calculate each successive value Y_{i+1} for $i = 0, 1 \ldots N - 1$ by approximating the integral of $y\prime(t)$ over $[t_i, t_{i+1}]$:

$$Y_{i+1} = Y_i + \int_{t_i}^{t_{i+1}} y'(t)dt$$

4. Calculate Y'_{i+1} using f and the approximate value for y_{i+1}:

$$Y'_{i+1} = f(t_{i+1}, Y_{i+1})$$

We can create different rules for solving the IVP by using different approximations of the definite integral

$$\int_{t_i}^{t_{i+1}} y'(t)dt$$

Last time we introduced two simple methods for solving the IVP, using the rectangle and midpoint rules:

Rule I: Euler's Method (Rectangle Rule) For each sub-interval, we have

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} y'(t)dt$$

Approximating this integral using the rectangle rule yields

$$Y_{i+1} = Y_i + h \cdot Y_i'$$

We can then calculate Y_{i+1}' using

$$Y_{i+1}' = f(t_{i+1}, Y_{i+1})$$

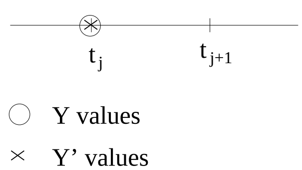

Figure 1: Function and Derivative Values used in Euler's Method

We say that this rule is single-step, since it only requires information about the current point t_i to calculate the values for the next point t_{i+1}, and no information about previous points is required. Figure 1 shows the function and derivative values required to apply Euler's rule.

Rule II: Modified Euler's Method (Midpoint Rule) In modified Euler's rule, we approximate the integral using the midpoint rule. However, recall that the midpoint rule requires that we know the value of the function we are integrating at the midpoint of the interval. To overcome this, we use an interval that is twice as large ($2h$), as shown in Figure 2.

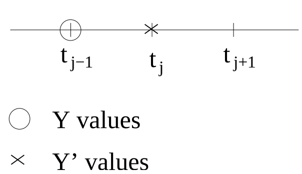

Figure 2: Function and Derivative Values used in Modified Euler's Method

$$
\begin{aligned}
y_{i+1} &= y_{i-1} + \int_{t_{i-1}}^{t_{i+1}} y'(t)dt \\
Y_{i+1} &= Y_{i-1} + 2h \cdot Y'_i
\end{aligned}
$$

We calculate the derivative value at t_{i+1} as always:

$$
Y'_{i+1} = f(t_{i+1}, Y_{i+1})
$$

We say that this rule is *multi-step*, since it requires knowledge of values at a past point (t_{i-1}) as well as the current point (t_i) to calculate the values at the next point (t_{i+1}). Although modified Euler's method is substantially better than Euler's method, it is not often used in practice since the error is still quite large compared to other methods.

Today we will study additional rules for solving the IVP, some of which are not used for approximating integrals in the general case, but are used for approximating integrals when solving the IVP.

2 Additional Rules for Solving IVP

2.1 Rule III: Trapezoid Method

Recall that, in the trapezoid rule, we approximate the integral using the line between the function values at the endpoints, so to approximate:

$$
y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} y'(t)dt
$$

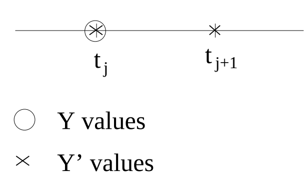

Figure 3: Function and Derivative Values used in the Trapezoid Method

we use the following approximation, as shown in Figure 3

$$
Y_{i+1} = Y_i + h \cdot \frac{(Y'_i + Y'_{i+1})}{2}
$$

Note that this method requires us to use Y'_{i+1}, which is a value that we do not know. Methods that require knowledge of the derivative at the point we are estimating are called *closed* methods. In general, to use a closed method, we must first use some other method to *predict* the value of

Y'_{i+1}. When we use an open method to predict the derivative value Y'_{i+1} and a closed method to use this value to calculate a corrected Y_i, we call such hybrid methods *predictor-corrector methods* and we will discuss them later in the lecture. The Trapezoid method is a *single step* method, since it requires no knowledge of function and derivative values at previous points.

2.2 Rule IV: Simpson's Method

Simpson's Method is a closed, multi-step method. As in the midpoint rule, we use an interval of size $2h$ for our approximation, as shown in Figure 4:

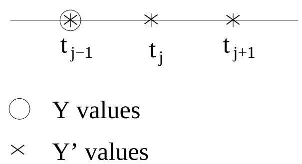

Figure 4: Function and Derivative Values used in Simpson's Method

The approximation for the new function value Y_{j+1} is given by:

$$Y_{j+1} \quad = \quad Y_{j-1} + \frac{h}{3}(Y'_{j-1} + Y'_j + Y'_{j+1})$$

2.3 Rule V: Milne's Method

Milne's method interpolates the function (that is $y'(t)$) using three equidistant points in the middle of the interval, as shown in Figure 5.

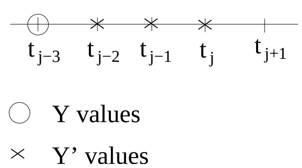

Figure 5: Function and Derivative Values used in Milne's Method

Milne's method is open and multi-step, using an interval of size $4h$ The approximation for the new function value Y_{j+1} is given by:

$$Y_{j+1} \quad = \quad Y_{j-3} + \frac{4h}{3}(2Y'_{j-2} - Y'_{j-1} + 2Y'_j)$$

2.4 Rule VI: Adams-Bashforth Method

The Adams-Bashforth rule is an open, multi-step method that uses four points to interpolate the function ($y'(t)$) being integrated, as shown in Figure 6. This method seems counter-intuitive, since it is asymmetric and uses many points off the interval to calculate the definite integral. Notes that

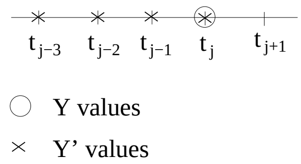

Figure 6: Function and Derivative Values used in the Adams-Bashforth Method

all of these values are available when we reach t_j, however so that the method is open. In practice, this rule does not suffer from many instabilities that many of the other rules share.

The approximation for the new function value Y_{j+1} in the Adams-Bashforth method is given by:

$$Y_{j+1} \quad = \quad Y_j + \frac{h}{24}(55Y'_j - 59Y'_{j-1} + 37Y'_{j-2} - 9Y'_{j-3})$$

2.5 Rule VII: Adams-Moulton Method

The Adams-Moulton rule is similar to Adams-Bashforth, but is closed, as shown in Figure 7

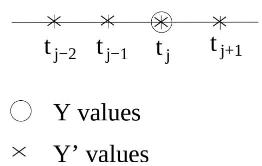

Figure 7: Function and Derivative Values used in the Adams-Moulton Method

The approximation for the new function value Y_{j+1} in the Adams-Moulton method is given by:

$$Y_{j+1} \quad = \quad Y_j + \frac{h}{24}(9Y'_{j+1} + 19Y'_j - 5Y'_{j-1} + Y'_{j-2})$$

2.6 Predictor-Corrector Methods

Note that, if we use an open method to *predict* a value Y_{j+1} (and associated derivative Y'_{j+1}), we can use this prediction as input to a closed rule to provide a (hopefully) better approximation.

We call these hybridized methods that pair together an open predictor rule with a closed correcting rule *predictor-corrector* methods. To calculate the next function value Y_{j+1} in a predictor-corrector method, we perform the following steps:

1. Use the (open) predictor method to calculate an approximation of the next function value, $\hat{Y}_{j+1}$.

2. Use the differential equation to find the approximate derivative at t_{j+1}

$$\hat{Y'}_{J+1} \quad = \quad f(t_{j+1}, \hat{Y}_{j+1})$$

3. Use the approximate derivative value you just calculated $Y_{j+1}^{\hat{\prime}}$ as input to the closed corrector method to get the final approximation Y_{j+1}.

For example, the Milne-Simpson predictor corrector uses Milne's rule as a predictor and Simpson's rule as a corrector.

The Adams-Bashforth-Moulton predictor corrector uses the Adams-Bashforth rule as predictor and the Adams-Moulton rule as corrector.

3 Example

Consider the following initial value problem:

$$\begin{aligned} y'(t) &= ty(t) \\ y(0) &= 1 \end{aligned}$$

It is easy to see that the analytical solution is

$$y(t) = e^{\frac{1}{2}t^2} \tag{1}$$

Let $h = 0.1$ and we want to compute $y(0.4)$. We will use Equation (1) to set $y(0.1), y(0.2)$ and $y(0.3)$:

$$\begin{aligned} Y_1 = y(0.1) = 1.0050125 &\qquad Y_1' = y'(0.1) = 0.10050125 \\ Y_2 = y(0.2) = 1.020201 &\qquad Y_1' = y'(0.2) = 0.2040402 \\ Y_3 = y(0.2) = 1.046028 &\qquad Y_1' = y'(0.3) = 0.3138084 \end{aligned}$$

And the correct value of $y(0.4) = 1.083287068$.

Euler's Method:

$$Y_4 = Y_3 + hY_3' = 1.077$$

Modified Euler's Method:

$$Y_4 = Y_2 + 2hY_3' = 1.08296$$

Trapezoid Rule:

$$Y_4 = Y_3 + \frac{h}{2}(Y_3' + Y_4')$$

Milne-Simpson predictor-corrector First we carry out Milne's rule as the predictor, since it is open

$$\hat{Y}_4 = Y_0 + \frac{4h}{3}(2Y_3' - Y_2' + 2Y_1') = 1.083275$$

We then use the differential equation to calculate $\hat{Y}'_4$, then use this as input to Simpson's rule (the corrector).

$$Y_4 = Y_2 + \frac{h}{3}(Y_2' + 4Y_3' + \hat{Y}_4') = 1.083285$$

which is very close to the correct value.