Lecture 23: Systems of Differential Equations

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1 Review

For the last few lectures, we have discussed the solution of initial value problems for first order differential equations. Recall that the initial value problem is a problem of the form:

PROBLEM 1.1. *Given*

$$\begin{aligned} y'(t) &= f(t, y(t)) \\ y(a) &= y_0 \end{aligned}$$

find $y(b)$.

We have discussed several methods for solving the IVP, all using the idea of splitting the interval $[a, b]$ into N subintervals using partition points $(t_0, t_1, \ldots, t_N)$, then approximating the definite integral of y' over each subinterval. In this lecture, we shall discuss the solution of higher order differential equations.

2 Higher Order Differential Equations

To begin, we consider second order differential equations. When solving second order differential equations we would like to find a function $y(t)$ given an expression for its second derivative in terms of t, $y(t)$, and $y'(t)$:

PROBLEM 2.1. *Given the second order differential equation:*

$$y''(t) = f(t, y(t), y'(t))$$

find $y(t)$.

As was the case with first order differential equations, this is not a well-defined problem. For example, consider the second order differential equation:

$$y''(t) = -y(t)$$

It is not difficult to see that functions like:

$$\begin{aligned} y(t) &= k_1 \cdot \sin(t) \\ y(t) &= k_2 \cdot \cos(t) \end{aligned}$$

are correct solutions. In fact, we can characterize all correct solutions using the linear combination.

$$y(t) = k_1 \cdot \sin(t) + k_2 \cdot \cos(t)$$

Note that this general solution has *two* parameters, and therefore we need two pieces of information to formulate the problem so that there is a unique solution. We will consider two correct formulations of the problem:

Initial Value Problem

In an initial value problem, we are given the values of y and y' at the same point a and want to find the value of y at a second point b.

PROBLEM 2.2. *Given:*

$$\begin{aligned} y''(t) &= f(t, y(t), y'(t)) \\ y(a) &= y_0 \\ y'(a) &= y'_0 \end{aligned}$$

find $y(b)$.

Boundary Value Problem

In a boundary value problem, we are given the value of y at two different points a and b, and want to find the value of y at some third point c, which is usually in the interval $[a, b]$.

PROBLEM 2.3. *Given:*

$$\begin{aligned} y''(t) &= f(t, y(t), y'(t)) \\ y(a) &= y_0 \\ y(b) &= y_1 \end{aligned}$$

find $y(c)$.

Solving boundary value problems is far more complex than solving initial value problems. We shall focus on initial value problems for the next two lectures.

(One could wonder: Does it make the problem well-posed to give the second derivative at a as our third piece of information? In other words, is the problem "*Given*

$$\begin{aligned} y''(t) &= f(t, y(t), y'(t)) \\ y(a) &= y_0 \\ y''(a) &= y''_0 \end{aligned}$$

find $y(c)$" well-posed? The answer is no. Either $y''(a)$ is redundant or it is contradictory. In our example above, suppose $y(0) = 1$. Then $y''(0) = -1$ is redundant with $y''(t) = -y(t)$, while $y''(0) = 2$ contradicts $y''(t) = -y(t)$. Similarly, giving y'(a point different from a) doesn't work.)

3 Solution of Second Order IVP

We use substantially the same method for solving a second order initial value problem that we used for solving a first order initial value problem. The trick is to reduce the second order IVP to a system of first order IVPs, which we then solve in tandem. Given a second order IVP:

PROBLEM 3.1. *Given:*

$$\begin{aligned} y''(t) &= f(t, y(t), y'(t)) \\ y(a) &= y_0 \\ y'(a) &= y'_0 \end{aligned}$$

find $y(b)$.

we first define two new functions $x_1(t)$ and $x_2(t)$:

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) = x'_1(t) \end{aligned}$$

We can then rewrite the second order differential equation from problem 3.1 as two first order differential equations:

$$\begin{aligned} x'_1(t) &= x_2(t) \\ x'_2(t) &= f(t, x_1(t), x_2(t)) \end{aligned}$$

Then we rewrite the initial values for y and y' in terms of x_1 and x_2:

$$\begin{aligned} x_1(a) &= y_0 \\ x_2(a) &= y'_0 \end{aligned}$$

As before, we solve the problem by splitting up the interval $[a, b]$ into N equal-sized partitions using partition points $(t_0, t_1, \ldots, t_N)$. We must approximate both functions on each subinterval, so we calculate two approximate values at each point t_i: $X_{1,i}$ and $X_{2,i}$, then use f to calculate $X'_{2,i}$. Note that $X'_{1,i} = X_{2,i}$ so that we only calculate three total values at each point. Here we define $X_{i,j}$ as our approximation of $x_i(t_j)$ and $X'_{i,j}$ as our approximation of $x'_i(t_j)$.

Now we can apply any method that we have studied for solving first order initial value problems to solve this problem. The only issue is that we must calculate all the approximations $X_{1,i}, X_{2,i}, X'_{2,i}$ at point t_i before we can calculate the approximations at the next point t_{i+1}, since the values are coupled (we use the X_2 values to approximate X_1, and we need both X_1 and X_2 values to calculate X'_1). To illustrate, consider the following example

EXAMPLE 3.1. *Given the second order IVP:*

$$\begin{aligned} y''(t) &= -ty(t) + y'(t)^2 \\ y(2) &= 5 \\ y'(2) &= -1 \end{aligned}$$

find $y(3)$.

Solution:

1. Define functions $x_1(t)$, $x_2(t)$:

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \end{aligned}$$

2. Rewrite as a system of first order differential equations

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -t \cdot x_1(t) + x_2(t)^2 \end{aligned}$$

3. Define initial values for the first partition point t_0:

$$\begin{aligned} X_{1,0} &= 5 \\ X_{2,0} &= -1 \\ X_{2,0}' &= -t \cdot X_{1,0} + X_{2,0}^2 \\ &= -(0)(5) + (-1)^2 \\ &= 1 \end{aligned}$$

4. Solve using any method for first order IVP

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4 Systems of First Order IVP

Note that this method for solving second order IVP is a special case of a general method for solving systems of first order IVP. Given the system of first order initial value problems:

$$\begin{aligned} x_1'(t) &= f_1(t, x_1(t), x_2(t)) \\ x_2'(t) &= f_2(t, x_1(t), x_2(t)) \\ x_1(a) &= x_{1,0} \\ x_2(a) &= x_{2,0} \end{aligned}$$

We can solve this system by calculating our approximations for the value of each function x_i at each partition point t_j together (i.e. $X_{i,j}$).

4.1 Adapting Euler's Method to Systems of First Order Differential Equations

Recall that, to find the approximation Y_{j+1} of $y(t_{j+1})$ with Euler's method, we use:

$$Y_{j+1} = Y_j + h \cdot Y_j'$$

To adapt this rule to systems of first order differential equations, we must calculate the two approximations $X_{1,j+1}, X_{2,j+1}$:

$$\begin{aligned} X_{1,j+1} &= X_{1,j} + h \cdot X'_{1,j} \\ X_{2,j+1} &= X_{2,j} + h \cdot X'_{2,j} \end{aligned}$$

Example.
Say, $h = .1, t_0 = 0$,

$$X_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\begin{aligned} X'_{1,0} = 1 \\ X'_{2,0} = 1 \end{aligned}$$

so

$$X'_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We use Euler:

$$\begin{aligned} X_{1,1} &= X_{1,0} + hX'_{1,0} \\ &= 0 + .1 \cdot 1 = .1 \\ X_{2,1} &= X_{2,0} + hX'_{2,0} \\ &= 1 + .1 \cdot 0 = 1 \end{aligned}$$

so

$$X_1 = \begin{bmatrix} .1 \\ 1 \end{bmatrix}.$$