Lecture 24: Systems of First Order Differential Equations

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1 Systems of First Order IVP

In the last lecture, we discussed the solution of systems of first order initial value problems. Recall that in a system of first order IVP, we must find two unknown functions $x_1(t)$ and $x_2(t)$, given an expression for their derivatives and their values at some point a. The derivatives $x_1'(t)$ and $x_2'(t)$ are given in terms of $t, x_1(t)$ and $x_2(t)$, i.e.,

$$\begin{aligned} x_1'(t) &= f_1(t, x_1(t), x_2(t)) \\ x_2'(t) &= f_2(t, x_1(t), x_2(t)) \\ x_1(a) &= x_1^a \\ x_2(a) &= x_2^a \end{aligned}$$

Note that the IVP gives the value of x_1, x_2 at the same point. The goal is to find $x_1(t)$ and $x_2(t)$, in particular, to compute $x_1(b)$ and $x_2(b)$ for $b \neq a$.

As before, we use the following procedure to approximate $x_1(b)$, $x_2(b)$:

1. partition $[a, b]$ into N equal length sub-intervals using $a = t_0, t_1, \cdots, t_N = b$. Denote $X_{i,j} \approx x_i(t_j)$ and $X'_{i,j} \approx x_i'(t_j)$ for $i \in \{1, 2\}$, $0 \leq j \leq N$.
2. Start with $X_{1,0} = x_1^a$, $X_{2,0} = x_2^a$, calculate $X'_{1,0}$ and $X'_{2,0}$ using f_1, f_2.
3. Use one of our methods for approximating the integral to calculate $X_{1,j+1}$ using the values of $X'_{1,i}$ $(i \leq j+1)$. Do the same for $X_{2,j+1}$ using the values of $X'_{2,i}$ $(i \leq j+1)$.
4. Calculate $X'_{1,j+1}$, $X'_{2,j+1}$ using

$$\begin{aligned} X'_{1,j+1} &= f_1(t_{j+1}, X_{1,j+1}, X_{2,j+1}) \\ X'_{2,j+1} &= f_2(t_{j+1}, X_{1,j+1}, X_{2,j+1}) \end{aligned}$$

5. Repeat steps 3 and 4 until we find $X_{1,N}$, $X_{2,N}$

For example, Table 1 shows the process of calculating function and derivative values at t_{j+1} using modified Euler's rule.

t_0	$\dots$	t_j	t_{j+1}
$X_{1,0}$	$\dots$	$X_{1,j}$	$X_{1,j+1} = X_{1,j-1} + 2h \cdot X'_{1,j}$
$X'_{1,0}$	$\dots$	$X'_{1,j}$	$X'_{1,j+1} = f_1(t_{j+1}, X_{1,j+1}, X_{2,j+1})$
$X_{2,0}$	$\dots$	$X_{2,j}$	$X_{2,j+1} = X_{2,j-1} + 2h \cdot X'_{2,j}$
$X'_{2,0}$	$\dots$	$X'_{2,j}$	$X'_{2,j+1} = f_2(t_{j+1}, X_{1,j+1}, X_{2,j+1})$

Table 1: Calculating Function and Derivative Values at t_{j+1} Using Modified Euler's Rule

2 Higher Order Differential Equations

As we saw in the last lecture, solving a second order initial value problem is simply a special case of solving a system of first order IVP. Given a second order IVP like:

$$\begin{aligned} y''(t) &= f(t, y(t), y'(t)) \\ y(a) &= y_0 \\ y'(a) &= y'_0 \end{aligned}$$

We reformulate the problem as a system of first order IVP, using $x_1(t) = y(t)$, $x_2(t) = y'(t)$:

$$\begin{aligned} x'_1(t) &= x_2(t) \\ x'_2(t) &= f(t, x_1(t), x_2(t)) \\ x_1(a) &= y_0 \\ x_2(a) &= y'_0 \end{aligned}$$

We can then solve this system using the same method. However, since $x'_1(t) = x_2(t)$, we need only calculate 3 values at each partition point. Table 2 shows the process of calculating function and derivative values at t_{j+1} using modified Euler's rule.

t_0	$\dots$	t_j	t_{j+1}
$X_{1,0}$	$\dots$	$X_{1,j}$	$X_{1,j+1} = X_{1,j-1} + 2h \cdot X_{2,j}$
$X_{2,0}$	$\dots$	$X_{2,j}$	$X_{2,j+1} = X_{2,j-1} + 2h \cdot X'_{2,j}$
$X'_{2,0}$	$\dots$	$X'_{2,j}$	$X'_{2,j+1} = f(t_{j+1}, X_{1,j+1}, X_{2,j+1})$

Table 2: Calculating Function and Derivative Values at t_{j+1} Using Modified Euler's Rule

3 Runge-Kutta Method of Order Four

We diverge from our discussion of higher order differential equations to discuss a matter left unresolved in our treatment of first order IVP. We have discussed several rules for solving first order IVP, but all of the better rules are multi-step rules, so that we must calculate several values before the rule can be used, and this can only be done with a single-step, open rule. However, the only single-step, open rule we have discussed is Euler's method and we know that Euler's method introduces unacceptable amounts of error. So , the question remains, how do we get started when solving an IVP? Runge-Kutta methods are often used to do this.

We will discuss the Runge-Kutta method of order four. The end result of this method will be to approximate the value of t_{j+1} using Simpson's rule over the interval $[t_j, t_{j+1}]$. However, since Simpson's rule requires function values at the midpoint and endpoint of the interval, we will need to approximate those values. The main trick in Runge-Kutta methods is in approximating these values in a way that minimizes the total error of the approximation of t_{j+1}.

More formally, we want to calculate $y(t_{j+1})$ using:

$$y(t_{j+1}) = y(t_j) + \int_{t_j}^{t_{j+1}} y'(t)dt$$

Using Simpson's rule, we approximate $y(t_{j+1})$ using:

$$Y_{j+1} = Y_j + \frac{h}{6}\left(Y'_j + 4Y'_{j+1/2} + Y'_{j+1}\right)$$

Note that $Y'_{j+1/2}$ denotes the approximate value of $y'(t)$ at the midpoint of the interval $[t_j, t_{j+1}]$. The question is, how do we approximate the values of $Y'_{j+1/2}$ and Y'_{j+1}?

The idea is to use approximations of the derivative of y to estimate the values of y. We will use three different approximations of the derivative, each with potentially large error, but the idea is that the values are calculated and used in such a way that the errors will cancel. This process is depicted in Figure 1

We begin by approximating the value of y at the midpoint $Y_{j+1/2}$, by extending a line with slope Y'_j from (t_j, Y_j) to $t_{j+1/2}$:

$$Y_{j+1/2} = Y_j + \frac{h}{2}Y'_j$$

We can then use the differential equation for our first estimate of $Y'_{j+1/2}$:

$$Y'_{j+1/2} = f(t_{j+1/2}, Y_{j+1/2})$$

Recall that there is a solution to the differential equation (that is a function) that passes through every point, so that we are estimating the derivative of y, using the derivative of one of these functions. Next, we find another estimate of the midpoint (called $\tilde{Y}_{j+1/2}$), again by extending a line from the left endpoint, but this time using the slope at the midpoint:

$$\tilde{Y}_{j+1/2} = Y_j + \frac{h}{2}Y'_{j+1/2}$$

We can then use the differential equation for another estimate of the derivative at the midpoint $\tilde{Y}'_{j+1/2}$:

$$\tilde{Y}'_{j+1/2} = f(t_{j+1/2}, \tilde{Y}_{j+1/2})$$

Finally, we use this last derivative value to estimate the value of y at the endpoint, $\tilde{Y}_{j+1}$. Once again, we extend a line from the left endpoint, with slope $\tilde{Y}'_{j+1/2}$ to get our estimate:

$$\tilde{Y}_{j+1} = Y_j + h \cdot \tilde{Y}'_{j+1/2}$$

As usual, we use the differential equation for our estimate of the derivative at the endpoint:

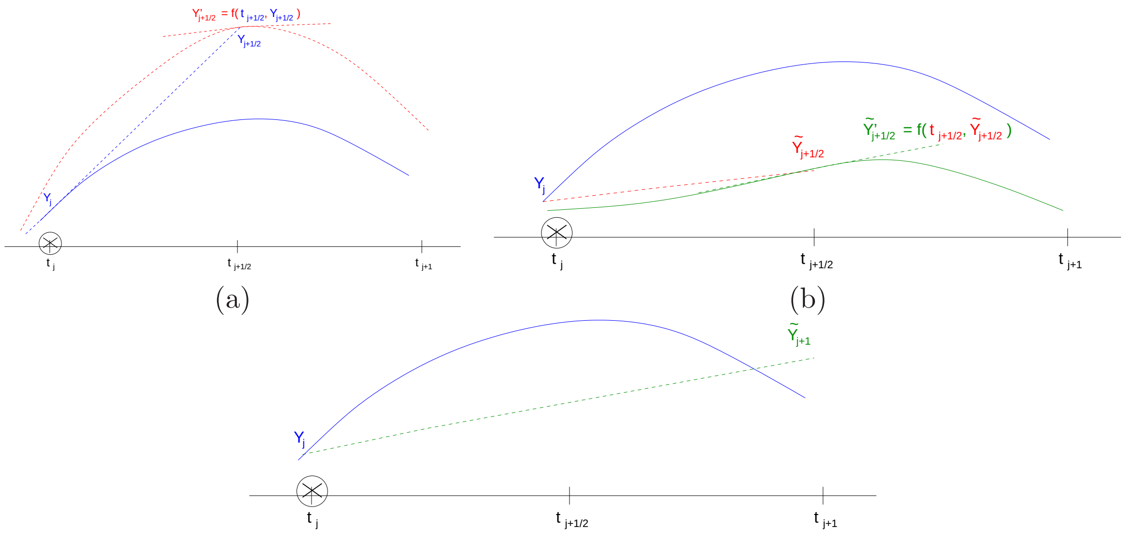
$$\tilde{Y}'_{j+1} = f(t_{j+1}, \tilde{Y}_{j+1})$$

It is important to note that all of these calculated values are simply used as parameters in Simpson's rule. Once we have used Simpson's rule to calculate our estimate of y at the endpoint (Y_{j+1}), we throw these values away. To get our final estimate of Y_{j+1}, we use Simpson's rule, with an average of the derivative values we estimated at the midpoint, and the value we estimated at the endpoint:

$$Y_{j+1} = Y_j + \frac{h}{6}\left(Y'_j + 4\frac{Y'_{j+1/2} + \tilde{Y}'_{j+1/2}}{2} + \tilde{Y}'_{j+1}\right) \tag{1}$$

$$= Y_j + \frac{h}{6}\left(Y'_j + 2Y'_{j+1/2} + 2\tilde{Y}'_{j+1/2} + \tilde{Y}'_{j+1}\right) \tag{2}$$

This method gives error of $O(h^4)$, so it can be used to start many of the higher-order multi-step methods.



(a) (b) (c)

Figure 1: Estimating Values at the Midpoint and Endpoint in the Runge-Kutta Method of Order 4 *(a) First, estimate the midpoint using the derivative at the left endpoint (b) Next, estimate the midpoint using the derivative calculated in step a (c) Lastly, estimate the right endpoint using the derivative value calculated in step b. Plug all these estimated values into Simpson's rule to obtain* Y_{j+1}.

Note that as for previous methods, one can use Runge-Kutta for a system of differential equations (and hence for higher-order differential equations). Here the only subtelty is that we need to march in parallel because we need all the Y values to evaluate the differential equation. For example, first do:

$$X_{1,j-1/2} = X_{1,j-1} + \frac{h}{2} X'_{1,j-1}$$
$$X_{2,j-1/2} = X_{2,j-1} + \frac{h}{2} X'_{2,j-1}$$

then do:

$$X'_{1,j-1/2} = f_1(t_{j-1} + \frac{h}{2}, X_{1,j-1/2}, X_{2,j-1/2})$$
$$X'_{2,j-1/2} = f_2(t_{j-1} + \frac{h}{2}, X_{1,j-1/2}, X_{2,j-1/2})$$