CS 525 - Spring 2011 -Midterm Examination
Tuesday, March 8, 2011, 2:30-3:45PM

1. For the following choice of A and b solve the system of equations $Ax = b$. If there are multiple solutions, describe the full solution set. If there are linear dependence relations between the rows of the coefficient matrix, state them.

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Solution:

First form a tableau

	x_1	x_2	x_3	1
y_1	1	-2	-1	-2
y_2	-1	-1	0	1

Swap x_3 and y_1:

	x_1	x_2	y_1	1
x_3	1	-2	-1	-2
y_2	-1	-1	0	1

Then swap x_1 and y_2:

	y_2	x_2	y_1	1
x_3	-1	-3	-1	-1
x_1	-1	-1	0	1

To read the tableau, we set y_1 and y_2 equal to zero. If we set $x_2 = t$, then we find the solution set equals

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \; : \; t \in \mathbb{R} \right\}.$$

2. Consider the following linear program

$$\begin{aligned} \text{minimize} \quad & 9x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 4 \\ & 3x_1 - x_2 \geq -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) Write down the dual of this problem.

(b) Find solutions for the primal and dual.

(c) Suppose the right-hand side of the first constraint is changed from 4 to 6. Without performing any additional simplex iterations or referring to the tableau, give a lower bound on the optimal primal objective value of the modified problem. Explain.

Solution:

(a) The dual problem is

$$\begin{aligned} \text{maximize} \quad & 4u_1 - 2u_2 \\ \text{subject to} \quad & u_1 + 3u_2 \leq 9 \\ & u_1 - u_2 \leq 1 \\ & u_1,\ u_2 \geq 0 \end{aligned}$$

(b) Observe that $(0, 0)$ is feasible for the dual. So we will solve this by the dual simplex method. Form the Tableau:

		u_3	u_4	
		x_1	x_2	1
$-u_1$	x_3	1	1	-4
$-u_2$	x_4	3	-1	2
		9	1	0

Swap u_1 and u_4

		u_3	u_1	
		x_1	x_3	1
$-u_4$	x_2	-1	1	4
$-u_2$	x_4	4	-1	-2
		8	1	4

And then swap u_2 and u_3

		u_2	u_1	
		x_4	x_3	1
$-u_4$	x_2	$-1/4$	$3/4$	$7/2$
$-u_3$	x_1	$1/4$	$1/4$	$1/2$
		2	3	8

As there are no pivots to improve the cost, we terminate at an optimal solution:

$$(\bar{x}_1, \bar{x}_2) = (1/2, 3/2), \qquad (\bar{u}_1, \bar{u}_2) = (3, 2)$$

(c) If we change the right hand side from 4 to 6, this changes the dual objective but not the dual constraints. Thus, our dual optimal solution $(3, 2)$ is feasible for the modified dual, and, by weak duality, provides us with the lower bound $6 * 3 - 2 * 2 = 14$ of the modified primal objective.

3. Consider the following linear program, where c_1, c_2, c_3 are constants:

$$\begin{array}{ll} \text{maximize} & c_1x_1 + c_2x_2 + c_3x_3 \\ \text{subject to} & -1 \leq x_1 \leq 1 \\ & -1 \leq x_2 \leq 1 \\ & -1 \leq x_3 \leq 1 \end{array}$$

(a) Write down the dual of this problem.

(b) Write down the KKT conditions for this problem.

(c) Find optimal solutions of the primal and dual problems that jointly satisfy the KKT conditions.

(d) Write the optimal cost of the primal problem solely in terms of the constants c_1, c_2, c_3.

Solution:

(a) Write the primal out in standard dual form:

$$\begin{array}{ll} \text{maximize} & c_1x_1 + c_2x_2 + c_3x_3 \\ \text{subject to} & x_1 \leq 1 \\ & -x_1 \leq 1 \\ & x_2 \leq 1 \\ & -x_2 \leq 1 \\ & x_3 \leq 1 \\ & -x_3 \leq 1 \end{array}$$

Then the dual is

$$\begin{array}{ll} \text{minimize} & u_1 + v_1 + u_2 + v_2 + u_3 + v_3 \\ \text{subject to} & u_1 - v_1 = c_1 \\ & u_2 - v_2 = c_2 \\ & u_3 - v_3 = c_3 \\ & u_1, u_2, u_3 \geq 0 \\ & v_1, v_2, v_3 \geq 0 \end{array}$$

(b) The KKT conditions are

$$\begin{array}{rrcrl} 0 \leq & 1 - x_1 & \perp & u_1 \geq & 0 \\ 0 \leq & 1 + x_1 & \perp & v_1 \geq & 0 \\ 0 \leq & 1 - x_2 & \perp & u_2 \geq & 0 \\ 0 \leq & 1 + x_2 & \perp & v_2 \geq & 0 \\ 0 \leq & 1 - x_3 & \perp & u_3 \geq & 0 \\ 0 \leq & 1 + x_3 & \perp & v_3 \geq & 0 \\ & u_1 - v_1 = & & c_1 & \\ & u_2 - v_2 = & & c_2 & \\ & u_3 - v_3 = & & c_3 & \end{array}$$

(c) If $c_i > 0$, set $u_i = c_i$, $v_i = 0$, and $x_i = 1$. If $c_i < 0$, set $u_i = 0$, $v_i = -c_i$, and $x_i = -1$. If $c_i = 0$, set $u_i = 0$, $v_i = 0$, and $x_i = 0$. It is readily verified that in all cases, the KKT conditions are satisfied.

(d) Using the dual optimal assignments, we see that $u_i + v_i = |c_i|$. Therefore, the optimal dual cost is equal to $|c_1| + |c_2| + |c_3|$. By strong duality, this is also equal to the optimal primal cost.