Exact recovery from ‘projections’ onto polynomial spaces over compact manifolds

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Case I: Trigonometric polynomials

- Spike train $\quad f(t)=\sum_{m=1}^{M} a_m \delta_{t_m}(t)$, $\quad \delta_x$ Dirac,

  $a_m \in \mathbb{C}$ coefficients, $-\pi \le t_1 < t_2 < \cdots < t_M < \pi$ knots.

- $\langle f, g \rangle = \sum_m a_m g(t_m), \qquad \forall g \in C[-\pi,\pi]$.
- f in dual space of Borel measures $\mathcal{M}([-\pi,\pi])$.
- Total variation of a complex measure over a compact $A \subset \mathbb{R}^n$

  $\|\mu\|_{TV} = |\mu|(A) := \sup \sum_k |\mu(A_k)|, \qquad A = \bigcup_k A_k$, interior disjoint finite collection.

- For a spike train $f(t)=\sum_m a_m \delta_{t_m}(t)$, $\|f\|_{TV}=\sum_m |a_m|$.

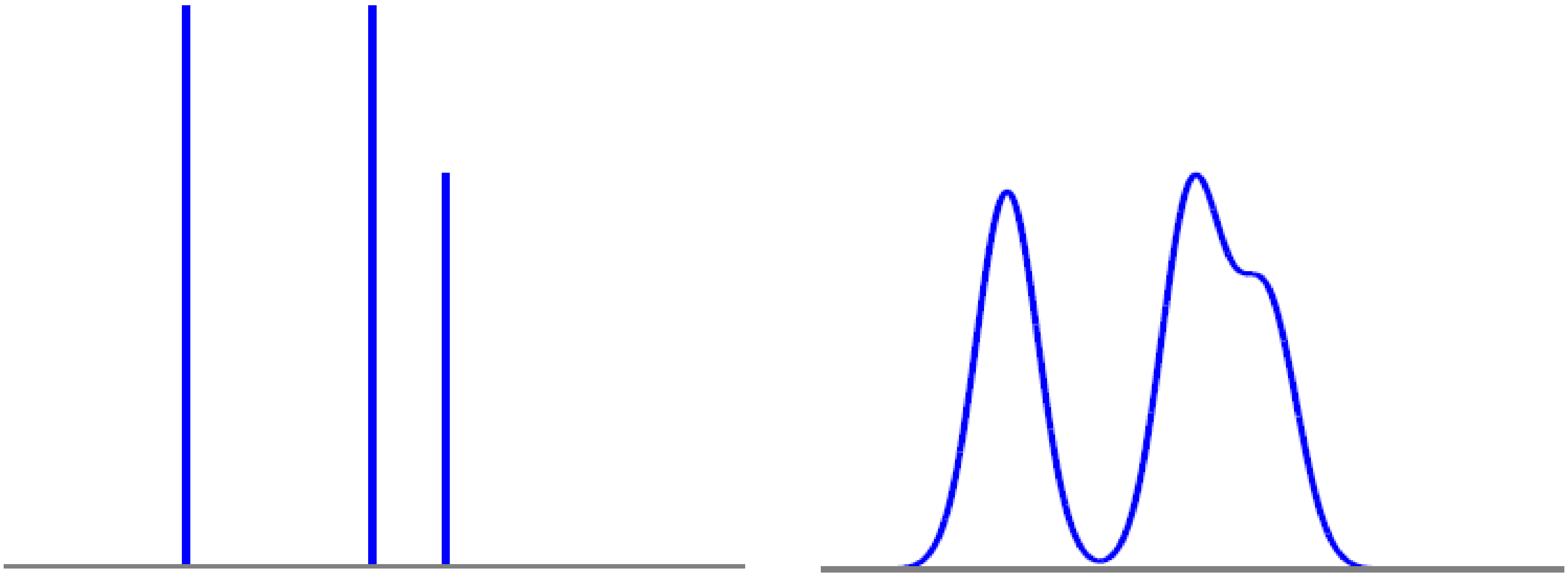
- **Input:** For some degree N, Fourier 'coefficients':

$$y_k := \langle f, e^{ik\cdot} \rangle = \frac{1}{2\pi}\sum_m a_m e^{-ikt_m}, \; -N \le k \le N.$$

Note: number of spikes and the coefficients are unknown.

- **Goal:** Recover f exactly from $\{y_k\}_{k=-N}^{N}$.

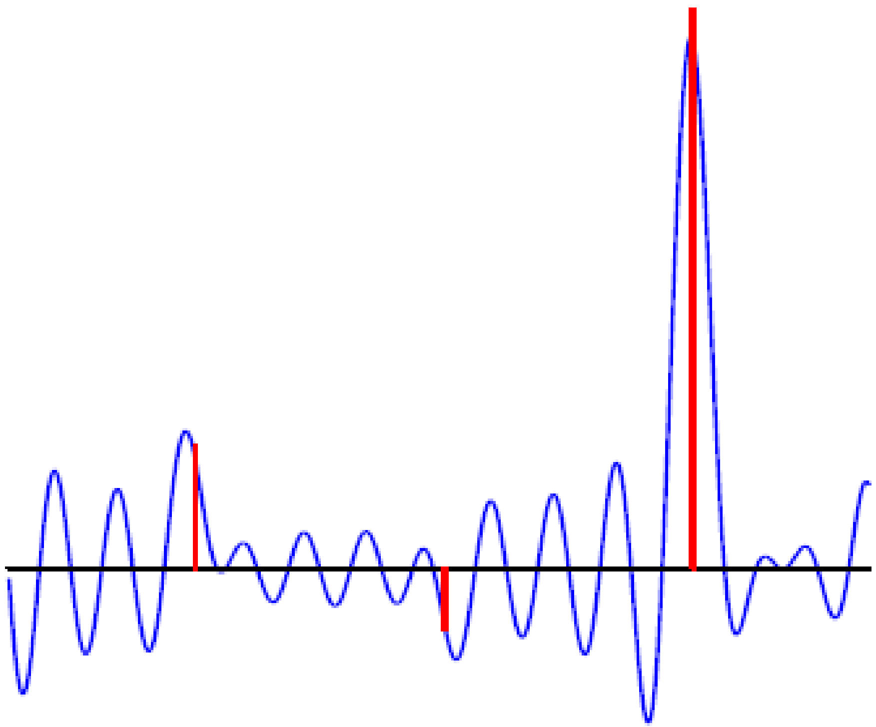
- E Candés & C. Fernandez-Granda, Towards a Mathematical Theory of Super-Resolution, 2012.

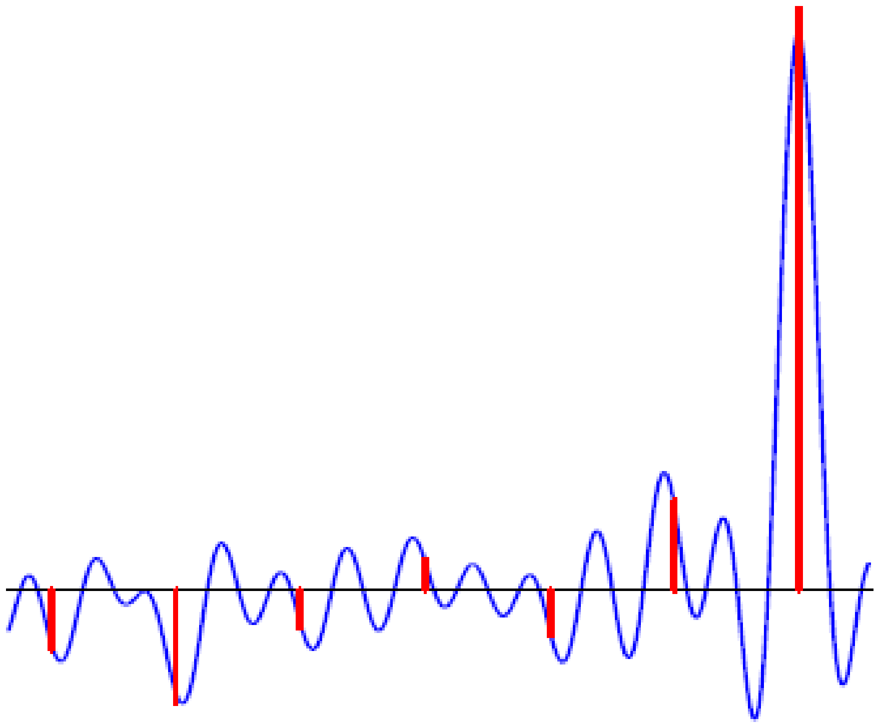


(a) (b)

Spikes and their lower-resolution 'projection'

(*) E Candés & C. Fernandez-Granda, Towards a Mathematical Theory of Super-Resolution, 2012.





Two different trains with qualitatively similar 'projections'

(*) E Candés & C. Fernandez-Granda, Towards a Mathematical Theory of Super-Resolution, 2012.

- **Separation condition [CF]** Assume that for $N \geq 128$

$$\Delta(T) := \min_{j,k} |t_j - t_k| \geq \frac{4\pi}{N} \quad \text{(Cyclic distance).}$$

Theorem [CF 2012] If the knots of a spike train f satisfy the separation condition and $y_k = \langle f, e^{ik\cdot} \rangle$, $-N \leq k \leq N$, are given, then f is the ***unique complex measure*** solving

$$\min_{\tilde{f} \in \mathcal{M}[-\pi,\pi]} \|\tilde{f}\|_{TV}, \qquad \text{s.t.} \left\langle \tilde{f}, e^{ik\cdot} \right\rangle = y_k , \quad -N \leq k \leq N.$$

- CF also analyzed the ‘applied setting’:
  (i) Resilience to noise,
  (ii) Stable recovery algorithm.

The 'dual interpolating polynomial'

Theorem 7. *Let* $f = \sum_m c_m \delta_{x_m}$ *where* $X := \{x_m\} \subseteq A$*, and* $A \subset \mathbb{R}^n$ *is compact. Let* Θ_D *be a linear space of continuous functions of dimension* $D+1$ *in* A*. For any basis* $\{\theta_k\}_{k=0}^{D}$*, of* Θ_D*, let* $y_k = \langle f, \theta_k \rangle$ *for all* $0 \leq k \leq D$*. If for any set* $\{u_m\}$*,* $u_m \in \mathbb{C}$*, with* $|u_m| = 1$*, there exists* $q \in \Theta_D$ *such that*

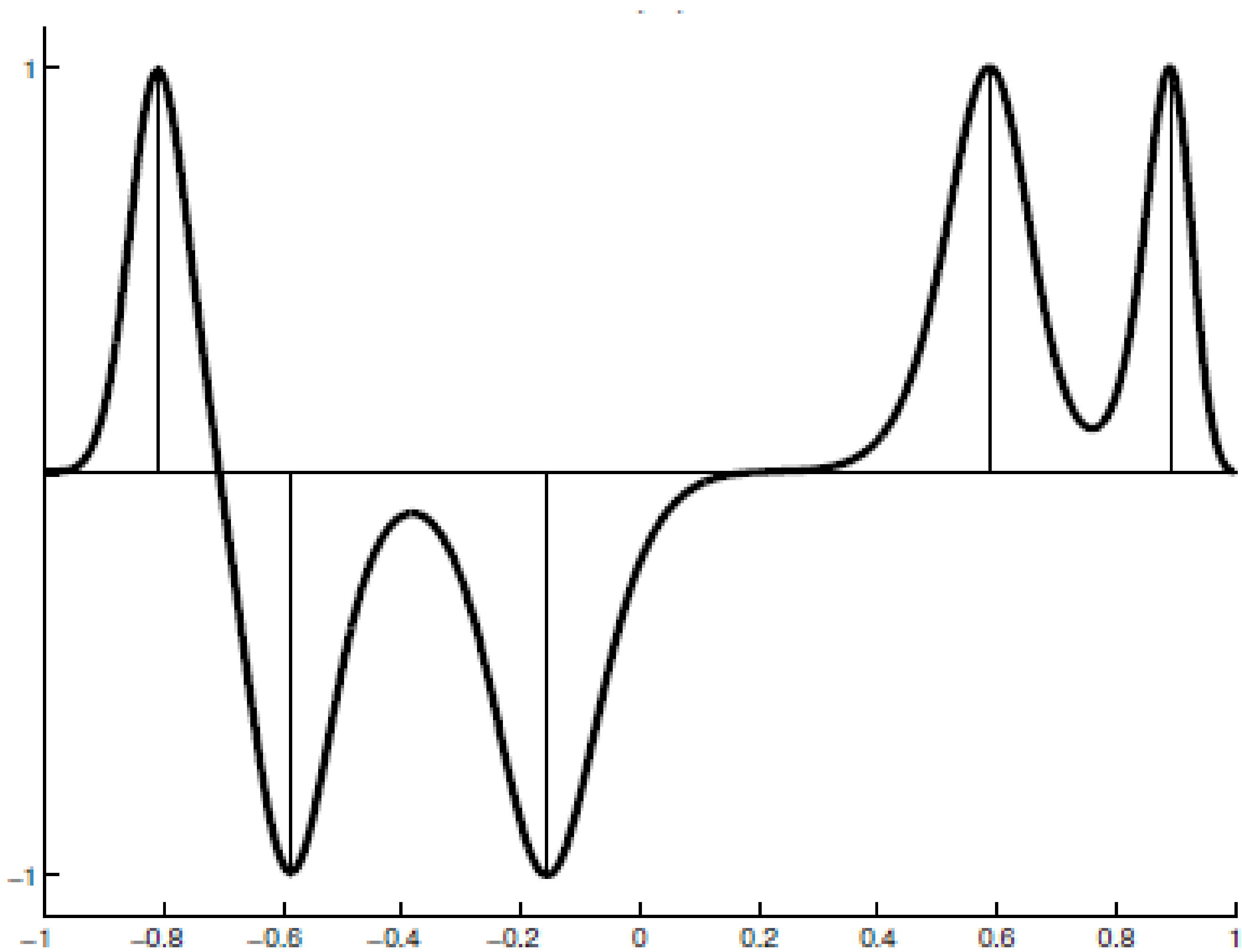
$$q(x_m) = u_m \,, \forall x_m \in X, \tag{2.1}$$

$$|q(x)| < 1 \,, \forall x \in A \backslash X, \tag{2.2}$$

then f *is the unique complex Borel measure satisfying*

$$\min_{g \in \mathcal{M}(A)} \|g\|_{TV} \quad \text{subject to} \quad y_k = \langle g, \theta_k \rangle \,, \, 0 \leq k \leq D. \tag{2.3}$$

Constructions of interpolating polynomials (with certain properties) $\Rightarrow$ Exact recovery of spike trains through TV-minimization.



An interpolating algebraic polynomial with values $u_m \in \{-1,1\}$ at specified knots

Non-negative signals

- Assume the unknown spike-train is of form

$$f(t)=\sum_{m=1}^{M} a_m \delta_{t_m}(t), \qquad a_m > 0, \quad t \in [-\pi,\pi].$$

Theorem If $M \le N$, then f is the unique minimizer over all non-negative real measures of

$$\min_{\tilde{f} \in \mathcal{M}_{\ge 0}[-\pi,\pi]} \|\tilde{f}\|_{TV}, \qquad \text{s.t. } \langle \tilde{f}, e^{ik\cdot} \rangle = y_k \ , \qquad -N \le k \le N.$$

Remark: Discrete 'non-negative' version known from Donoho-Tanner (2005). Very different arguments.

Here, we use a simple version of the ‘duality’ theorem for non-negative measures (the polynomial needs to be real).

The construction of q:

$$q(t) := 1 - 2^{-M-1} \prod_{m=1}^{M} \left(1 - \cos\left(t - t_m\right)\right)$$

q is a real trigonometric polynomial that satisfies

(i) q is a polynomial of degree $\leq N$.

(ii) $q(t_m) = 1$, for $1 \leq m \leq M$.

(iii) $0 < q(t) < 1$, for $t \neq t_m$, $1 \leq m \leq M$.

Case II: Algebraic polynomials

- Spike train $f(x) = \sum_m a_m \delta_{x_m}(x)$, δ_t Dirac,

  $a_m \in \mathbb{C}$ coefficients, $-1 \le x_1 < x_2 < \cdots < x_M < 1$ knots.

- **Input:** For some degree N and polynomial basis $\{P_k\}$ of V_N :

$$y_k := \langle f, P_k \rangle = \sum_m a_m P_k(x_m),\ 0 \le k \le N.$$

Recall: number of spikes and the coefficients are unknown.

- **Goal:** Recover f exactly from $\{y_k\}_{k=-N}^{N}$.

- **Separation condition** Assume for $N \geq 128$, that the knots satisfy

$$\frac{x_{k+1} - x_k}{\sqrt{1-\bar{x}^2}} \geq \frac{4\pi}{N}, \quad \bar{x} := \arg \min_{z \in [x_k, x_{k+1}]} |z|,$$

(+ another technical condition)

- Condition aligns with the classical metric over the interval. We can allow closer knots near the endpoints.

Theorem If the knots of a spike train f satisfy the (algebraic) separation condition and $y_k = \langle f, P_k \rangle$, $0 \leq k \leq N$, are given, then f is the ***unique complex measure*** solving

$$\min_{\tilde{f} \in \mathcal{M}[-1,1]} \|\tilde{f}\|_{TV}, \qquad \text{s.t. } \langle \tilde{f}, P_k \rangle = y_k, \qquad 0 \leq k \leq N.$$

Algebraic polynomials over $[-1,1]^2$

- Spike train $f(x)=\sum_m a_m \delta_{x_m}(x)$, δ_t Dirac,

$$a_m \in \mathbb{R} \text{ coefficients}, \quad x_m = \left(x_m(1), x_m(2)\right) \in (-1,1)^2 .$$

- **2D separation condition** Assume for $N \geq 128$, that

$$\min_{j,k}\left\{\frac{\left|x_k(1)-x_j(1)\right|}{\sqrt{1-\bar{x}_1^{\,2}}},\frac{\left|x_k(2)-x_j(2)\right|}{\sqrt{1-\bar{x}_2^{\,2}}}\right\} \geq \frac{4\pi}{N} ,$$

(+ technical condition).

Theorem If the knots of a 2D spike train f satisfy the 2D separation condition and $y_k = \langle f, P_k \rangle$, $k = (k_1, k_2)$, $0 \le k_1, k_2 \le N$, are given, then f is the ***unique real measure*** solving

$$\min_{\tilde{f} \in \mathcal{M}[-1,1]^2} \| \tilde{f} \|_{TV}, \qquad \text{s.t.} \left\langle \tilde{f}, P_k \right\rangle = y_k \ .$$

Spline case

Assume the unknown f is a piecewise constant (order $r=1$)

$$f(t)=c_0\mathbf{1}_{[-1,t_1)}+\sum_{m=1}^{M-1}c_m\mathbf{1}_{[t_m,t_{m+1})}(t)+c_M\mathbf{1}_{[t_M,1]}(t),$$

With known:

(i) boundary conditions $f(-1)=c_0$, $f(1)=c_M$ (M unknown!).

(ii) $y_k=\langle f,P_k\rangle$, $\{P_k\}$ some polynomial basis of V_N.

The distributional derivative is a spike train

$$f'(t)=\sum_{m=1}^{M-1}(c_m-c_{m-1})\delta_{t_m}(t).$$

What are $\{\langle f',P_k\rangle\}=?$

- Let $\{\alpha_{k,n}\}$ coefficients such that

$$P'_k = \sum_n \alpha_{k,n} P_n, \quad \forall k \ .$$

- Then with $y_k = \langle f, P_k \rangle$, integration by parts gives

$$\langle f', P_k \rangle = f(1) P_k(1) - f(-1) P_k(-1) - \sum_n \alpha_{k,n} y_n \ .$$

- Exact recovery of the spike train f' yields exact recovery of the piecewise constant f.

- Generalization to spline of arbitrary order r (degree $r-1$):
  - Assume boundary conditions $f^{(j)}(-1), f^{(j)}(1)$, $j = 0, \dots, r-1$, are known.
  - Recover (via recursion) f from the spike train $f^{(r)}$.

Case III: Spherical Harmonics on $\mathbb{S}^{d-1}$

- $\mathbb{Y}_n\left(\mathbb{R}^d\right)$ Homogeneous Harmonic polynomials of degree n .
- $\mathbb{Y}_n^d := \mathbb{Y}_n\left(\mathbb{R}^d\right)\Big|_{\mathbb{S}^{d-1}}$ Spherical Harmonics of degree n in d dimensions.
- Generalization of trigonometric polynomials.
- Let $\left\{P_{n,k}\right\}$, $n \le N$, $1 \le k \le Z_{n,d}$ be a basis for spherical harmonics of degree N

$$Z_{n,d} = \frac{(2n+d-2)(n+d-3)!}{n!(d-2)!}$$

$$f(\xi) = \sum_m a_m \delta_{\xi_m}(\xi), \qquad a_m \in \mathbb{R}\ ,\ \Xi = \{\xi_m\}\ ,\ \xi_m \in \mathbb{S}^2 \subset \mathbb{R}^3.$$

- **Sphere separation condition** For a fixed constant $\nu > 1$, assume the points $\Xi = \{\xi_m\}$ satisfy for (sufficiently large) N,

$$\min_{\xi_j, \xi_k \in \Xi} \operatorname{arcos}\left(\xi_j \cdot \xi_k\right) \geq \frac{\nu}{N}.$$

Theorem If the set $\Xi = \{\xi_m\}$ satisfies the separation condition for sufficiently large N and $y_{n,k} = \langle f, P_{n,k} \rangle$, $n \leq N$, are given, then f is the ***unique real measure*** solving

$$\min_{\tilde{f} \in \mathcal{M}(\mathbb{S}^2)} \left\| \tilde{f} \right\|_{TV}, \quad \text{s.t.}\ \langle \tilde{f}, P_{n,k} \rangle = y_{n,k}\ .$$

So what's under the hood? A localization principle!

Localization principle

- Recall that we need to construct q, such that

  (i) $q(t_m) = u_m$, t_m knots, $u_m \in \mathbb{C}$, $|u_m| = 1$ prescribed.

  (ii) $|q(t)| < 1$, $t \neq t_m$.

- Proof relies on finding well-localized polynomial kernel $K_N(x, y)$ of the given degree N.
- Trigonometric polynomials

$$q(t) = \sum_m \left(\alpha_m K(t - t_m) + \beta_m K'(t - t_m) \right),$$

$K_N(x, y) = K(x - y)$, the Jackson kernel, $\{\alpha_m\}, \{\beta_m\}$ selected to satisfy conditions ($q(t_m) = u_m$, …).

- Algebraic polynomials
  - Well localized kernel exists, but not translation invariant!
  - Direct proof exists…but easier to reduce the problem to the trigonometric case.

- Spherical harmonics
  - Rotation invariant well-localized(!) kernel,

$$K_N(\xi_1 \cdot \xi_2) = \kappa_N \sum_{n=0}^{\infty} \varphi\left(\frac{n}{N}\right) \mathcal{P}_n(\xi_1 \cdot \xi_2),$$

$\mathcal{P}_n(\xi_1 \cdot \xi_2)$ - ortho-projection onto harmonics of degree n,

$$\varphi \in C^{\infty}(\mathbb{R}_+),\ \varphi(t) = \begin{cases} 1 & t \le 1/2, \\ 0 \le \varphi(t) \le 1 & 1/2 \le t < 1, \\ 0 & else. \end{cases}$$

$$\left|K_N\left(\xi_1\cdot\xi_2\right)\right| \le \frac{c_k}{\left(1+N\ \operatorname{arcos}\left(\xi_1\cdot\xi_2\right)\right)^k}, \qquad \xi_1,\xi_2\in\mathbb{S}^{d-1}.$$

- Lie algebra structure - rotational derivatives.
- With ν the constant from the separation condition, we prove estimates of the type

$$\left\|I-\left(K_N\left(\xi_j\cdot\xi_k\right)\right)_{j,k}\right\|_\infty \le \frac{c_k}{\nu^{k-1}}.$$