Accuracy and Stability: recent advances in C.A.G.D.

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Effects of finite precision arithmetic on numerical algorithms:

- Roundoff errors.
- Data uncertainty.

Key concepts:

- *Conditioning*: it measures the sensibility of solutions to perturbations of data.
- *Growth factor*: it measures the relative size of the intermediate computed numbers with respect to the initial coefficients or to the final solution.
- *Backward error*: if the computed solution is the exact solution of a perturbated problem, it measures such perturbation.
- *Forward error*: it measures the distance between the exact solution and the computed solution.

(Forward error) $\leq$ *(Backward error)* $\times$ *(Condition)*

Growth factor

$$\rho_n^W(A) := \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|}$$

ρ_n^W associated with partial pivoting of an $n \times n$ matrix is bounded above by 2^n. ρ_n^W associated with complete pivoting of an $n \times n$ matrix is "usually" bounded above by n.

Gauss elimination of a symmetric positive definite matrix (without row or column exchanges) presents $\rho_n^W = 1$.

Amodio and Mazzia have introduced the growth factor

$$\rho_n(A) := \frac{\max_k \|A^{(k)}\|_\infty}{\|A\|_\infty}.$$

P. Amodio, F. Mazzia: A new approach to backward error analysis of LU factorization, BIT 39 (1999) pp. 385–402.

Condition number

$$\kappa(A) := \|A\|_\infty \, \|A^{-1}\|_\infty.$$

The Skeel condition number:

$$\mathrm{Cond}(A) := \| \, |A^{-1}| \, |A| \, \|_\infty.$$

- $\mathrm{Cond}(A) \le \kappa(A)$
- $\mathrm{Cond}(DA) = \mathrm{Cond}(A)$ for any nonsingular diagonal matrix D

Accurate calculation: the relative error is bounded by $\mathcal{O}(\varepsilon)$, where ε is the machine precision.

Admissible operations in algorithms with high relative precision: products, quotients, sums of numbers of the same sign and sums/subtractions of exact data:

The only **forbidden** operation is true subtraction, due to possible cancellation in leading digits.

A nonsingular matrix A with positive diagonal elements and nonpositive off-diagonal elements is an **M-matrix** if $A^{-1} \geq 0$.

A matrix is **totally positive** if all its minors are nonnegative.

In order to guarantee accurate computations for some special classes of matrices, it is crucial to find an **adequate parametrization** of the special classes of matrices:

- For diagonally dominant M-matrices: the off-diagonal entries and the row sums.
- For nonsingular totally positive matrices: the multipliers of its Neville elimination.

basis $u = (u_0, \ldots, u_n)$ of a real vector space $\mathcal{U}$ of functions defined on a subset K of $\mathbf{R}^s$ and a function $f \in \mathcal{U}$,

$$f(x) = \sum_{i=0}^{n} c_i u_i(x).$$

We want to know how sensitive a value $f(x)$ is to any perturbations of a given maximal relative magnitude ε in the coefficients $c_0, \ldots, c_n$ corresponding to the basis. The corresponding perturbation $\delta f(x)$ of the change of $f(x)$ can be bounded by means of a **condition number**

$$C_u(f, x) := \sum_{i=0}^{n} |c_i u_i(x)|,$$

for the evaluation of $f(x)$ in the basis u:

$$|\delta f(x)| \le C_u(f(x))\varepsilon.$$

R. T. Farouki & V. T. Rajan (1988): On the numerical condition of polynomials of algebraic curves and surfaces 1. Implicit equations. *Comput. Aided Geom. Design* **5**, 215-252.

Farouki, R. T. & Goodman, T. N. T. (1996): On the optimal stability of Bernstein basis. *Math. Comp.* **65**, 1553–1566.

Relative condition number:

$$c_u(f,x) := \frac{C_u(f,x)}{|f(x)|} \left(= \frac{\sum_{i=0}^n |c_i u_i(x)|}{|\sum_{i=0}^n c_i u_i(x)|} \right).$$

Let $\hat{f}(x)$ be the computed value wit floating point arithmetic.

$$\hat{f}(x) = \sum_{i=0}^{n} \bar{c}_i u_i(x).$$

Backward error analysis provides bounds for

$$\frac{|\bar{c}_i - c_i|}{|c_i|} \quad \text{or} \quad |\bar{c}_i - c_i|$$

Forward error analysis provides bounds for

$$|f(x) - \hat{f}(x)|$$

(Forward error) ≤ (Backward error) × (Condition)

The natural partial order for real-valued functions induces a corresponding partial order on the bases for $\mathcal{U}$, via

$$u \preceq v \quad \text{if and only if} \quad C_u(f,t) \leq C_v(f,t), \ \forall f \in \mathcal{U}, \ \forall t \in I.$$

Given a set $\mathcal{B}$ of bases of bases of a vector space $\mathcal{U}$ of functions defined on I, we say that a basis $b \in \mathcal{B}$ is **optimally stable** for the evaluation of functions among all bases of $\mathcal{B}$ if it is minimal with respect to this partial order among all bases in $\mathcal{B}$. We shall consider the set $\mathcal{B}$ of bases of $\mathcal{U}$ formed by functions with constant sign (i.e., each basis function is either nonnegative or nonpositive).

Theorem. The normalized B-bases are optimally stable.

Extension of optimally stable bases beyond total positivity context.

For spaces of **univariate** functions:

P. J.M.: "On the optimal stability of bases of univariate functions" (2002). *Numerische Mathematik* **91**, pp. 305-318.

P. J.M.: "A note on the optimal stability of bases of univariate functions" (2006). *Numerische Mathematik* **103**, pp. 151-154.

For spaces of **multivariate** functions:

LYCHE T., P. J.M.: "Optimally stable multivariate bases" (2004). *Advances in Computational Mathematics* **20**, pp. 149-159.

The tensor product b^{mn} of Bernstein bases is **optimally stable** on $[0, 1] \times [0, 1]$.

The tensor product of B-splines bases is **optimally stable**.

The Bernstein basis B of multivariate polynomials defined on a triangle (resp., tetrahedron) is **optimally stable**.

The error analysis of the corresponding evaluation algorithms performed in:

MAINAR E., P. J.M.: "Running error analysis of evaluation algorithms for bivariate polynomials in barycentric Bernstein form" (2006). *Computing* **77**, 97-111.

MAINAR E., P. J.M.: "Evaluation algorithms for multivariate polynomials in Bernstein Bézier form" (2006). *Journal of Approximation Theory* **143**, 44-61.

DELGADO J., P. J.M.: "Error analysis of efficient evaluation algorithms for tensor product surfaces" (2008). *Journal of Computational and Applied Mathematics* **219**, pp. 156-169.

Rational Bézier surfaces

Given the double-index $\alpha = \alpha_1\alpha_2$, with $0 \le \alpha_1 \le m$, $0 \le \alpha_2 \le n$, we can define the corresponding basis function

$$r_\alpha(x,y) := \frac{w_\alpha\, b^m_{\alpha_1}(x)\, b^n_{\alpha_2}(y)}{\sum_{\alpha_1=0}^{m}\sum_{\alpha_2=0}^{n} w_\alpha\, b^m_{\alpha_1}(x)\, b^n_{\alpha_2}(y)}.$$

The previous basis is **optimally stable**.

DELGADO J., P. J.M.: "A Corner Cutting Algorithm for Evaluating Rational Bézier Surfaces and the Optimal Stability of the Basis" (2007). *SIAM J. Scient. Comput.* **29**, pp. 1668-1682.

The usual method to evaluate rational Bézier surfaces uses the projection operator. In contrast, we propose a new evaluation method such that *all* steps are convex combinations. It is a **robust** algorithm with optimal growth factor.

Both previous algorithms are more stable than evaluation algorithms of nested type and with lower complexity which have also been considered.

We have also analyzed the running error analysis of the projection and the new evaluation algorithm. A posteriori error bounds are calculated simultaneously with the evaluation algorithm without increasing the computational cost considerably.

DELGADO J., P. J.M.: "Running Relative Error for the Evaluation of Polynomials" (2009). *SIAM Journal on Scientific Computing* **31** , pp. 3905-3921.

DELGADO J., P. J.M.: "Running error for the evaluation of rational Bézier surfaces" (2010). *Journal of Computational and Applied Mathematics* **233**, pp. 1685-1696.

DELGADO J., P. J.M.: "Running error for the evaluation of rational Bézier surfaces through a robust algorithm". *Journal of Computational and Applied Mathematics* (2011). *Journal of Computational and Applied Mathematics* **235**, pp. 1781-1789.

Are the rational tensor product systems derived from the Bernstein basis **monotonicity preserving**?

The answer is negative even for $m = n = 1$.

DELGADO J., P. J.M.: "Are rational Bézier surfaces monotonicity preserving?" (2007). *Computer Aided Geometric Design*. **24**, pp. 303-306.

Triangular rational Bézier surfaces and monotonicity preservation

The Bernstein polynomials of degree n on a triangle, $(b_{\mathbf{i}}^n)_{|\mathbf{i}|=n}$ are defined by $b_{\mathbf{i}}^n(\tau) = \frac{n!}{i_0!i_1!i_2!}\,\tau_0^{i_0}\,\tau_1^{i_1}\,\tau_2^{i_2}$, $|\mathbf{i}| = n$.

Now let us consider the rational Bernstein basis of order n $(\phi_{\mathbf{i}})_{|\mathbf{i}|=n}$ given by $\phi_{\mathbf{i}} = \frac{w_{\mathbf{i}}\,b_{\mathbf{i}}^n}{\sum_{|\mathbf{i}|=n} w_{\mathbf{i}}\,b_{\mathbf{i}}^n}$, where $(w_{\mathbf{i}})_{|\mathbf{i}|=n}$ is a sequence of positive weights.

The previous basis is **optimally stable**.

DELGADO J., P. J.M.: "On the evaluation of rational triangular Bézier surfaces and the optimal stability of the basis" (2013). *Advances in Computational Mathematics* **13**, pp. 701-721.

Is the rational Bernstein basis **axially monotonicity preserving**?

The answer is negative up to the polynomial case.

THEOREM. If a rational Bernstein basis on a triangle with positive weights is axially monotonicity preserving, then $w_{\mathbf{i}} = w_{\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j}$ such that $|\mathbf{i}| = |\mathbf{j}| = n$.

Mixed spaces

$$U_n = \langle 1, \ldots, t^{n-2}, \cosh(wt), \sinh(wt) \rangle$$
$$U_n = \langle 1, \ldots, t^{n-2}, \cos(wt), \sin(wt) \rangle$$

MAINAR E., P. J.M.: "Optimal Bases for a class of mixed spaces and their associated spline spaces" (2010). *Computers and Mathematics with Applications* **59**, pp. 1509-1523.

The normalized basis $(u_{0,n}, \ldots, u_{n,n})$ of U_n can be defined by:

$$u_{0,n}(t) := 1 - \int_0^t \delta_{0,n-1} u_{0,n-1}(s) ds,$$

$$u_{i,n}(t) := \int_0^t \left(\delta_{i-1,n-1} u_{i-1,n-1}(s) - \delta_{i,n-1} u_{i,n-1}(s)\right) ds,\ i = 1, \ldots, n-1,$$

$$u_{n,n}(t) := \int_0^t \delta_{n-1,n-1} u_{n-1,n-1}(s) ds, \quad \delta_{i,n-1} := 1 / \int_0^a u_{i,n-1}(s)\, ds$$

Optimal stability and shape preserving properties

Theorem. Let $(u_{0,m},\ldots,u_{m,m})$ and $(v_{0,n},\ldots,v_{n,n})$, $m>1$, $n>1$, be two bases defined by the previous recurrences on the intervals $[0,a_1]$ and $[0,a_2]$. The tensor product blending system

$$(u_{i,m}\otimes v_{j,n})_{i=0,\ldots,m,\, j=0,\ldots,n}$$

defined on $[0,a_1]\times[0,a_2]$ **preserves axial monotonicity**.

MAINAR E., P. J.M.: "Monotonicity preserving representations of non-polynomial surfaces" (2010). *Journal of Computational and Applied Mathematics* **233**, 2161-2169.

Theorem. Let $(u_{0,m},\dots,u_{m,m})$ and $(v_{0,n},\dots,v_{n,n})$, $m>1$, $n>1$, be two bases defined by the previous recurrences on the intervals $[0,a_1]$ and $[0,a_2]$. The tensor product basis $(u_{i,m}\otimes v_{j,n})_{i=0,\dots,m,\,j=0,\dots,n}$ defined on $[0,a_1]\times[0,a_2]$ **preserves monotonicity** with respect to the abscissae $(\nu_0,\dots,\nu_m)$ and $(\tau_0,\dots,\tau_n)$ such that

$$\nu_0=0,\quad \nu_{i+1}-\nu_i=\int_0^{a_1} u_{i,m-1}(s)\,ds,\quad i=0,\dots,m-1$$
$$\tau_0=0,\quad \tau_{j+1}-\tau_j=\int_0^{a_2} v_{j,n-1}(s)\,ds,\quad j=0,\dots,n-1.$$

On the **critical length** of spaces U_n:

CARNICER J.M., MAINAR E., P. J.M.: "On the critical lengths of cycloidal spaces". To appear in *Constructive Approximation*.

Almost accurate polynomial evaluation in:

J.M. Carnicer, T.N.T. Goodman, J.M. P.: "Roundoff errors for polynomial evaluation by a family of formulae" (2008). *Computing* **82**, pp. 199-215.

Recurrences and **evaluation** algorithms for **multivariate orthogonal polynomials** in:

BARRIO R., P. J.M., SAUER T.: "Three term recurrence for the evaluation of multivariate orthogonal polynomials" (2010). *Journal of Approximation Theory* **162**, pp. 407-420.

Accurate SVDs of diagonally dominant matrices and of some TP matrices

A *rank revealing decomposition* of a matrix A is a decomposition $A = XDY^T$, where X, Y are well conditioned and D is a diagonal matrix. In that paper it is shown that if we can compute an accurate rank revealing decomposition then we also can compute (with an algorithm presented there) an accurate singular value decomposition. Obviously, an LDU-factorization with L, U well conditioned, is a rank revealing decomposition.

J. Demmel, M. Gu, S. Eisenstat, I. Slapnicar, K. Veselic and Z. Drmac: Computing the singular value decomposition with high relative accuracy, Linear Algebra Appl. **299** (1999), 21-80.

They provided an algorithm for computing an **accurate singular value decomposition from a rank revealing decomposition** with a complexity of $\mathcal{O}(n^3)$ elementary operations.

Accurate SVDs of diagonally dominant matrices

J. Demmel and P.S. Koev: Accurate SVDs of weakly dominant M-matrices, *Numer. Math.* **98** (2004), pp. 99-104.

They present a method to compute accurately an LDU-decomposition of an $n \times n$ M**-matrix diagonally dominant by rows**. They use symmetric complete pivoting and so they can guarantee that one of the obtained triangular matrices is diagonally dominant and the other one has the off-diagonal elements with absolute value bounded above by the diagonal element

J.M. P.: LDU decompositions with L and U well conditioned". *Electronic Transactions of Numerical Analysis* **18**, pp. 198-208.

The m.a.d.d. pivoting strategy is used and so **both** triangular matrices are **diagonally dominant**.

With a low computational cost:

BARRERAS A., P. J.M.: "Accurate and efficient LDU decompositions of diagonally dominant M-matrices" (2012). *Electronic Journal of Linear Algebra* **24**, pp. 153-167.

General diagonally dominant matrices in:

Q. Ye, Computing singular values of diagonally dominant matrices to high relative accuracy, *Math. Comp.* **77** (2008), 2195-2230.

TP and SR matrices

Definition. A matrix is *strictly totally positive* (STP) if all its minors are positive and it is *totally positive* (TP) if all its minors are nonnegative.

Definition. A matrix is called *sign-regular* (SR) if all $k \times k$ minors of A have the same sign (which may depend on k) for all k. If, in addition, all minors are nonzero, then it is called *strictly sign-regular* (SSR).

Variation diminishing properties of sign-regular matrices A: if A is a nonsingular $(n+1) \times (n+1)$ matrix, then A is sign-regular if and only if the number of changes of strict sign in the ordered sequence of components of $A\mathbf{x}$ is less than or equal to the number of changes of strict sign in the ordered sequence $(x_0, \ldots, x_n)$, for all $\mathbf{x} = (x_0, \ldots, x_n)^T \in \mathbf{R}^{n+1}$.

Proposition. Let A be a nonsingular TP matrix. Then all the eigenvalues of A are positive.

Nice properties of eigenvalues and eigenvectors of these matrices

Factorizations in terms of bidiagonal matrices

If K is TP and nonsingular, then we can write

$$K = L_{n-1}L_{n-2}\cdots L_1 D U_1 \cdots U_{n-2}U_{n-1},$$

where the matrices L_i (resp., U_i) are nonnegative lower (resp., upper) triangular **bidiagonal** with unit diagonal and D is a **diagonal** matrix with **positive** diagonals.

Uniqueness of the factorization under certain conditions in:

M. Gasca, J.M. P.: A matricial description of Neville elimination with applications to total positivity. *Linear Alg. Appl.* **202** (1994), 33–54.

Pinkus, Allan: *Totally positive matrices.* Cambridge Tracts in Mathematics, 181. Cambridge University Press, Cambridge, 2010.

If K is SR and nonsingular, then we can write

$$K = L_{n-1} L_{n-2} \cdots L_1 D U_1 \cdots U_{n-2} U_{n-1},$$

where the matrices L_i (resp., U_i) are nonnegative lower (resp., upper) triangular **bidiagonal** with unit diagonal and D is a **diagonal** matrix with nonzero diagonals:

M. Gasca, J.M. P.: A test for strict sign-regularity. *Linear Alg. Appl.* **197-198** (1994), 133–142.

If K is nonsingular TP and stochastic, then we can write

$$K = F_{n-1} F_{n-2} \cdots F_1 G_1 \cdots G_{n-2} G_{n-1},$$

with

$$F_i = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & \alpha_{i+1,1} & 1-\alpha_{i+1,1} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \alpha_{n,n-i} & 1-\alpha_{n,n-i} \end{pmatrix}$$

and

$$G_i = \begin{pmatrix} 1 & 0 & & & & & \\ & \ddots & \ddots & & & & \\ & & 1 & 0 & & & \\ & & & 1-\alpha_{1,i+1} & \alpha_{1,i+1} & & \\ & & & & \ddots & \ddots & \\ & & & & & 1-\alpha_{n-i,n} & \alpha_{n-i,n} \\ & & & & & & 1 \end{pmatrix},$$

where, $\forall\,(i,j)$, $0 \le \alpha_{i,j} < 1$.

Interpretation in **CAGD** of this factorization as a **corner cutting algorithm**: the most important algorithms in CAGD.

The bidiagonal factorization of nonsingular TP matrices is associated to an elimination procedure alternative to Gauss elimination called **Neville elimination**. It requires $O(n^3)$ elementary operations to check if an $n \times n$ matrix is either TP or STP:

M. Gasca, J.M. P.: Total positivity and Neville elimination. *Linear Algebra Appl.* **165** (1992), 25-44.

Neville elimination produces zeros in each column by adding to each row an adequate multiple of the previous one (instead of a multiple of the pivot row as in Gauss elimination).

Neville elimination leads to a factorization of a nonsingular totally positive matrix in terms of **bidiagonal** factors, and the elements appearing in the factorization are natural parameters of the matrix.

This factorization has been used to obtain **accurate computations** with nonsingular totally positive matrices. In particular, accurate computation of their SVDs and eigenvalues.

P. Koev: Accurate computations with totally nonnegative matrices, *SIAM J. Matrix Anal. Appl.* **29** (2007), no. 3, 731–751.

P. Koev: Accurate Eigenvalues and SVDs of Totally Nonnegative Matrices, *SIAM J. Matrix Anal. Appl.* **27** (2005), 1-23.

J. Demmel and P. Koev: The Accurate and Efficient Solution of a Totally Positive Generalized Vandermonde Linear System, *SIAM J. Matrix Anal. Appl.* **27** (2005), 142-152.

J.J. Martínez, J.M. P.: Fast algorithms of Bjrck-Pereyra type for solving Cauchy-Vandermonde linear systems, *Appl. Numer. Math.* **26** (1998), 343-352.

J.J. Martínez, J.M. P.: Factorizations of Cauchy-Vandermonde matrices, *Linear Algebra Appl.* **284** (1998), 229–237.

A. Marco, J.J. Martínez: A fast and accurate algorithm for solving Bernstein-Vandermonde linear systems, *Linear Algebra Appl.* **422** (2007), 616-628.

A. Marco, J.J. Martínez: Accurate computations with Said-Ball-Vandermonde matrices, *Linear Algebra Appl.* **432** (2010), 2894-2908.

A. Marco, J.J. Martínez: Polynomial least squares fitting in the Bernstein basis, *Linear Algebra Appl.* 433 (2010), 1254-1264.

J. Delgado, J.M. P.: Accurate computations with collocation matrices of rational bases (2013). *Applied Mathematics and Computation* **219**, pp. 4354-4364.

A more general class of matrices with bidiagonal decomposition and accurate computations

$$L^{(k)} = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & l_{n-k}^{(k)} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & l_{n-1}^{(k)} & 1 \end{pmatrix}$$

$$U^{(k)} = \begin{pmatrix} 1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ & & & 1 & u_{n-k}^{(k)} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & 1 & u_{n-1}^{(k)} \\ & & & & & & 1 \end{pmatrix}$$

Let A be a nonsingular $n \times n$ matrix. Suppose that we can write A as a product of bidiagonal matrices

$$A = L^{(1)} \cdots L^{(n-1)} D U^{(n-1)} \cdots U^{(1)}, \tag{1}$$

with

1. $d_i \neq 0$ for all i,
2. $l_i^{(k)} = u_i^{(k)} = 0$ for $i < n-k$,
3. $l_i^{(k)} = 0 \Rightarrow l_{i+s}^{(k-s)} = 0$ for $s = 1, \ldots, k-1$ and $u_i^{(k)} = 0 \Rightarrow u_{i+s}^{(k-s)} = 0$ for $s = 1, \ldots, k-1$.

Then we denote (1) by $\mathcal{BD}(\mathcal{A})$, a bidiagonal decomposition of A satisfying the conditions of this definition.

Theorem. Let A be a nonsingular matrix. If a $\mathcal{BD}(\mathcal{A})$ exists, then it is unique.

Let us denote by ε the vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$ with $\varepsilon_j \in \{\pm 1\}$ for $j = 1, \ldots, m$, which will be called a *signature*.

Given a signature $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1})$ and a nonsingular $n \times n$ matrix A, we say that A has a **signed bidiagonal decomposition with signature** ε if there exists a $\mathcal{BD}(\mathcal{A})$ such that

$d_i > 0$ for all i,

$l_i^{(k)} \varepsilon_i \geq 0$, $u_i^{(k)} \varepsilon_i \geq 0$ for $1 \leq k \leq n-1$ and $n-k \leq i \leq n-1$.

Totally positive matrices and their inverses are particular examples.

BARRERAS A., P. J.M.: "Accurate computations of matrices with bidiagonal decomposition using methods for totally positive matrices" (2013). *Numerical Linear Algebra with Applications* **20**, pp. 413-424..

Definition. A system of functions $(u_0, \ldots, u_n)$ is *totally positive* (TP) if all its collocation matrices are totally positive.

TP systems of functions are interesting due to the *variation diminishing* properties of totally positive matrices

Definition. A TP basis $(u_0, \ldots, u_n)$ is *normalized totally positive* (NTP) if

$$\sum_{i=0}^{n} u_i(t) = 1, \quad \forall t \in I.$$

Collocation matrices of NTP systems are TP and stochastic

In CAGD, NTP bases are associated with **shape preserving representations**.

The **normalized B-basis** is the basis with **optimal shape preserving properties**.

The **Bernstein** basis is the normalized B-basis of the space of polynomials of degree less than or equal to n on a compact interval $[a, b]$:

$$b_i(t) := \binom{n}{i} \left(\frac{t-a}{b-a}\right)^i \left(\frac{b-t}{b-a}\right)^{n-i}, \quad i = 0, \ldots, n.$$

CARNICER J.M., P. J.M.: "Shape preserving representations and optimality of the Bernstein basis" (1993). *Advances in Computational Mathematics* **1**, pp. 173-196.

CARNICER J.M., P. J.M.: "Totally positive bases for shape preserving curve design and optimality of B-splines" (1994). *Computer Aided Geometric Design* **11**, pp. 633-654.

Minimal eigenvalue of TP matrices

DELGADO J., P. J.M.: "Progressive iterative approximation and bases with the fastest convergence rates" (2007). *Computer Aided Geometric Design* **24**, pp. 10-18.

Theorem. The minimal eigenvalue of a Bernstein collocation matrix is always greater than or equal to the minimal eigenvalue of the corresponding collocation matrix of another NTP basis of the space.

P. J.M.: "Eigenvalue bounds for some classes of P-matrices". *Numerical Linear Algebra with Applications* **16** (2009), pp. 871-882.

Given $i \in \{1, \ldots, n\}$ let

$$J_i := \{j \mid |j-i| \text{ is odd}\}, \quad K_i := \{j \neq i \mid |j-i| \text{ is even}\}.$$

Theorem. Let A be a nonsingular totally positive matrix, and let $\lambda_{\min}(> 0)$ be its minimal eigenvalue. Then:

$$\lambda_{\min} \geq \min_i \{a_{ii} - \sum_{j \in J_i} a_{ij}\}. \tag{1}$$

Gerschgorin Theorem applied to any real matrix $A = (a_{ij})_{1 \le i,j \le n}$ implies that

$$\min_i \{a_{ii} - \sum_{j \neq i} a_{ij}\} \le \min_i \{\mathrm{Re}\lambda_i\}. \tag{2}$$

The following nonsingular matrix A is totally positive:

$$A = \begin{pmatrix} 12 & 7 & 1 \\ 0 & 6 & 1 \\ 0 & 3 & 8 \end{pmatrix}.$$

The eigenvalues of A are 12, 9 and 5. The bound given by (1) implies that $\lambda_{\min} \ge 5$ and so it **cannot be improved**. However, the lower bound for $\lambda_{\min}$ given by (2) is now $\lambda_{\min} \ge \min\{4, 5, 5\} = 4$.

Optimal conditioning of Bernstein collocation matrices

DELGADO J., P. J.M.: "Optimal conditioning of Bernstein collocation matrices" (2009). *SIAM J. Matrix Anal. Appl.* **31**, 990-996.

Theorem. Let $(b_0, \ldots, b_n)$ be the **Bernstein basis**, let $(v_0, \ldots, v_n)$ be another NTP basis of P_n on $[0, 1]$, let $0 \leq t_0 < t_1 < \cdots < t_n \leq 1$ and $V := M\binom{v_0,\ldots,v_n}{t_0,\ldots,t_n}$ and $B := M\binom{b_0,\ldots,b_n}{t_0,\ldots,t_n}$. Then

$$\kappa_\infty(B) \leq \kappa_\infty(V).$$

$$\mathrm{Cond}(A) := \| \, |A^{-1}| \, |A| \, \|_\infty .$$

Theorem. Let $(b_0, \ldots, b_n)$ be the **Bernstein basis**, let $(v_0, \ldots, v_n)$ be another TP basis of P_n on $[0, 1]$, let $0 \le t_0 < t_1 < \cdots < t_n \le 1$ and $V := M\begin{pmatrix} v_0, \ldots, v_n \\ t_0, \ldots, t_n \end{pmatrix}$ and $B := M\begin{pmatrix} b_0, \ldots, b_n \\ t_0, \ldots, t_n \end{pmatrix}$. Then

$$\mathrm{Cond}(B^T) \le \mathrm{Cond}(V^T).$$

ALONSO P., DELGADO J., GALLEGO R., P. J.M. (2013): "Conditioning and accurate computations with Pascal matrices". *Journal of Computational and Applied Mathematics* **252**, pp. 21-26.

Progressive iterative approximation

H. LIN, H. BAO, G. WANG (2005), Totally positive bases and progressive iteration approximation, Computer & Mathematics with Applications 50, 575-586.

$$\gamma(t) = \sum_{i=0}^{m} P_i\, u_i(t)$$

Now we parameterize the control points P_i with the real increasing sequence $t_0 < t_1 < \cdots < t_m$, where the parameter t_i is assigned to the control point P_i for $i = 0, 1, \ldots, m$. Then we have the starting curve

$$\gamma^0(t) = \sum_{i=0}^{m} P_i^0\, u_i(t)$$

of the sequence where $P_i^0 = P_i$ for $i = 0, 1, \ldots, m$. The remaining curves of the sequence, $\gamma^{k+1}(t)$ for $k \geq 0$, can be calculated as follows:

$$\gamma^{k+1}(t) = \sum_{i=0}^{m} P_i^{k+1} u_i(t),$$

with $\Delta_i^k = P_i - \gamma^k(t_i)$ and $P_i^{k+1} = P_i^k + \Delta_i^k$ or $i = 0, 1, \ldots, m$. Then $\Delta_j^k = \Delta_j^{k-1} - \sum_{i=0}^{n} \Delta_i^{k-1} u_i(t_j)$, for $j = 0, 1, \ldots, m$.

The iterative process can be written in **matrix form** in the following way:

$$\left[\Delta_0^k, \Delta_1^k, \ldots, \Delta_m^k\right]^\top = (I - B)\left[\Delta_0^{k-1}, \Delta_1^{k-1}, \ldots, \Delta_m^{k-1}\right]^\top$$

where I is the identity matrix of $n+1$ order and B is the collocation matrix of the basis $(u_0, \ldots, u_m)$ at $t_0 < t_1 < \cdots < t_m$.

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Theorem. The progressive iterative approximation process converges for any nonsingular collocation matrix B of an NTP basis.

Key fact: $\rho(I - B) < 1$ (B has positive eigenvalues because it is totally positive)

Which are the bases with fastest convergence rates?

Theorem. Given a space U with an NTP basis, the normalized B-basis of U provides a progressive iterative approximation with the fastest convergence rates among all NTP bases of U.

Theorem. Given spaces U, V with NTP bases, the tensor product of the normalized B-bases of U, V provides a progressive iterative approximation with the fastest convergence rates among all bases which are tensor product of NTP bases of U, V.

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