# Convergence of uniform subdivision

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Cascade and subdivision defined

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#### mask and dilation

 $a \in \mathbb{C}^{\mathbb{Z}^d/2}$  is a finite mask. Considered as a discrete finite measure  $\mathcal{D}$  is dyadic dilation:

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#### Question

Given a compactly supported g, do we have

$$\|\boldsymbol{C}^{k}\boldsymbol{g}-\phi\|_{\infty} \to 0?$$

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Each of the following conditions is necessary:

• 
$$\boldsymbol{g}, \phi \in \boldsymbol{C}^{\alpha}$$
,  $\alpha \geq 0$ .

• 
$$\sum_{j\in\gamma+2\mathbb{Z}^d} a(j) = 1, \gamma \in \{0,1\}^d.$$

- $g \phi$  has zero mean.
- The PSI space S(g) provides approximation order 1 in the  $\infty$ -norm, viz., for each sufficiently smooth *f*, as  $k \to \infty$ ,

$$\operatorname{dist}_{L_{\infty}}(f, \mathcal{D}^{k}S(g)) = O(2^{-k}).$$

## $G_0$

is the collection of compact support *g* that satisfy the above.

## Subdivision: definition and convergence

## Definition: the space $Q_k$

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#### Definition: The subdivision operator $S_k$ , convergence

$$S_k: Q_0 \to Q_k, \quad \lambda \mapsto \mathcal{D}^{k-1}a * S_{k-1}\lambda.$$

Convergence:

$$\|\mathcal{D}^k g * S_k \delta - \phi\|_{\infty} \to 0, \quad \forall g \in G_0.$$

# Subdivision: definition and convergence

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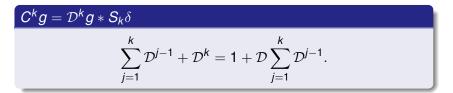
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#### The Transfer operator T

With *f* a trig. pol., and  $\tau := |\hat{a}|^2$ ,

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The transfer operator encodes  $L_2$ -properties of *a* and  $\phi$ , including a complete characterization of the convergence of cascade: essentially it need to have a unique dominant eigenvalue (acting on any large enough set of trig. pol.).

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The transfer operator also encodes the  $L_2$ -smoothness of  $\phi$ .

# The $L_2$ -case is spectral

#### The Transfer operator *T*

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The transfer cannot be used (obvious reasons) for other norms.

## Characterization: joint spectral radius

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Despite of its name, the joint spectral radius is joint but not spectral.

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Convergence of cascade: (more or less) we need that

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#### Special case: box splines, de Boor-R

If  $\phi$  is a box spline, then spectral.

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