Multivariate polynomial interpolation on lower sets

Nira Dyn and Michael Floater

Department of Mathematics,
University of Oslo

Lower set interpolation

J. Kuntzmann, *Méthodes numériques: interpolation, dérivées*, 1959.

H. Werner, Remarks on Newton type multivariate interpolation for subsets of grids, 1980.

G. G. Lorentz and R. A. Lorentz, Solvability problems of bivariate interpolation I, 1986.

G. Mühlbach, On multivariate interpolation by generalized polynomials on subsets of grids, 1988.

C. de Boor and A. Ron, On multivariate interpolation, 1990.

C. de Boor, On the error in multivariate polynomial interpolation, 1992.

M. Gasca and T. Sauer, Polynomial interpolation in several variables, 2000.

T. Sauer, Lagrange interpolation on subgrids of tensor product grids, 2004.

A. Chkifa, A. Cohen, and C. Schwab, High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs, 2013.

Sparse grids

S.A. Smolyak, Quadrature and interpolation formulas for tensor products of certain classes of functions, 1963.
V. Barthelmann, E. Novak, and K. Ritter, High dimensional polynomial interpolation on sparse grids, 2000.
M. Hegland, The combination technique and some generalisations 2007.

Cartesian grid of points

Cartesian grid of points in $\mathbb{R}^d$,

$$x_\alpha = (x_{1,\alpha_1}, x_{2,\alpha_2}, \ldots, x_{d,\alpha_d}), \qquad \alpha \in \mathbb{N}_0^d,$$

where $x_{j,k}$, $k \in \mathbb{N}_0$, are distinct for each $j \in \{1, \ldots, d\}$.
Multi-index notation:

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d,$$

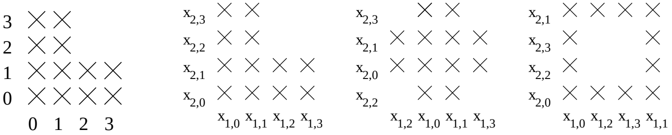
with $|\alpha| := \alpha_1 + \cdots + \alpha_d$, and $\alpha \le \beta$ means that $\alpha_j \le \beta_j$ for $j = 1, \ldots, d$.

Lower sets

We call a finite set $L \subset \mathbb{N}_0^d$ a *lower set* if whenever $\mu \in L$ and $0 \leq \alpha \leq \mu$ then $\alpha \in L$. Let

$$Y_L = \{x_\alpha : \alpha \in L\}.$$

The set Y_L can take on different configurations.



Interpolation on lower sets

Let

$$P_L = \mathrm{span}\{x^\alpha : \alpha \in L\},$$

where x^α is the monomial

$$x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}.$$

For every function f defined on Y_L there is a unique polynomial $p \in P_L$ that interpolates f on Y_L, i.e., such that $p(x_\alpha) = f(x_\alpha)$ for all $\alpha \in L$.

Earliest reference: J. Kuntzmann, 1959.

The Newton form

One way of expressing p is in Newton form. For each $j = 1, \ldots, d$, let $\omega_{j,0}(y) = 1$ and

$$\omega_{j,k}(y) = \prod_{i=0}^{k-1}(y - x_{j,i}), \qquad k \geq 1, \quad y \in \mathbb{R},$$

and define the d-variate polynomial

$$\omega_\alpha(x) = \omega_{1,\alpha_1}(x_1) \cdots \omega_{d,\alpha_d}(x_d), \qquad \alpha \in \mathbb{N}_0^d.$$

Let $\Delta_{\alpha,\beta} f$, $0 \leq \alpha \leq \beta$, be the the tensor-product divided difference of f over the points μ, $\alpha \leq \mu \leq \beta$. Then we can express p as

$$p(x) = \sum_{\alpha \in L} \omega_\alpha(x) \Delta_{0,\alpha} f, \qquad x \in \mathbb{R}^d. \tag{1}$$

We will sometimes write p as $p(L)$.

Interpolation in terms of blocks

A point $\beta \in L$ is a *maximal* point if there is no $\mu \in L$ such that $\beta \neq \mu$ and $\beta \leq \mu$. Let $V \subset L$ be the set of maximal points. Then

$$L = \bigcup_{\beta \in V} B_\beta,$$

where B_β is the (rectangular) 'block'

$$B_\beta = \{\alpha \in \mathbb{N}_0^d : 0 \leq \alpha \leq \beta\},$$

For example, in the figure,

$$L = B_{1,3} \cup B_{3,1}.$$

Interpolation in terms of blocks

Suppose first that L is the union of two blocks:

$$L = B_\alpha \cup B_\beta.$$

Then

$$p(L) = p(B_\alpha) + p(B_\beta) - p(B_\alpha \cap B_\beta).$$

Proof. Use the Newton form of $p(L)$. Since

$$p(L)(x) = \sum_{\mu \in B_\alpha \cup B_\beta} \omega_\mu(x) \Delta_{0,\mu} f, \qquad x \in \mathbb{R}^d,$$

the result follows from the fact that

$$\sum_{\alpha \in L} = \sum_{\alpha \in B_\alpha} + \sum_{\alpha \in B_\beta} - \sum_{\alpha \in B_\alpha \cap B_\beta}.$$

Arbitrary number of blocks

Similarly, if L is any lower set and B a block,

$$p(L \cup B) = p(L) + p(B) - p(L \cap B).$$

Therefore, if

$$L_r = B_1 \cup B_2 \cup \cdots \cup B_r,$$

then

$$p(L_n) = p(L_{n-1}) + p(B_n) - p(L_{n-1} \cap B_n),$$

and we obtain the double sum formula

$$p(L_n) = \sum_{i=1}^{n} p(B_i) - \sum_{i=2}^{n} p(L_{i-1} \cap B_i).$$

Two dimensions

Suppose $B_1, \ldots, B_n$ are blocks, $B_i = B_{\beta^i}$, in $\mathbb{R}^2$. We can order them so that

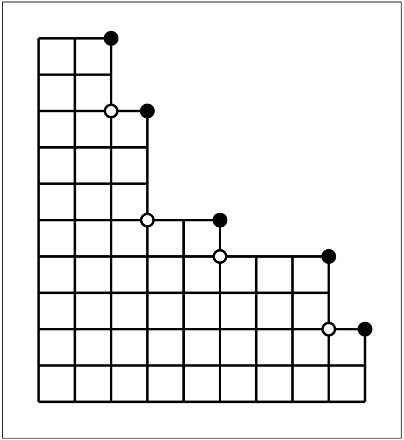
$$0 \leq \beta_1^1 < \beta_1^2 < \cdots < \beta_1^n, \qquad \beta_2^1 > \beta_2^2 > \cdots > \beta_2^n \geq 0.$$

The blocks form a staircase. The double sum formula simplifies to

$$p(L_n) = \sum_{i=1}^{n} p(B_i) - \sum_{i=2}^{n} p(B_{i-1} \cap B_i).$$

Example, with $n=5$

The points β^i are black circles, The points $(\beta_1^{i-1}, \beta_2^i)$ are white circles.



Arbitrary dimension

With the shorthand

$$p_{i_1,\ldots,i_k} := p(B_{i_1} \cap \cdots \cap B_{i_k}),$$

repeated use of the double sum formula leads to

Theorem

$$p(L_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} p_{i_1,\ldots,i_k}.$$

The first few cases are

$$\begin{aligned}
p(L_2) &= (p_1 + p_2) - p_{12}, \\
p(L_3) &= (p_1 + p_2 + p_3) - (p_{12} + p_{13} + p_{23}) + p_{123}, \\
p(L_4) &= (p_1 + p_2 + p_3 + p_4) - (p_{12} + p_{13} + p_{14} + p_{23} + p_{24} + p_{34}) \\
&\quad + (p_{123} + p_{124} + p_{134} + p_{234}) - p_{1234}.
\end{aligned}$$

How can we simplify in arbitrary dimension?

The theorem implies there are integer coefficients c_α, $\alpha \in L$, such that

$$p(L) = \sum_{\alpha \in L} c_\alpha p(B_\alpha).$$

Let $\chi(L) : \mathbb{N}_0^d \to \{0, 1\}$ be the characteristic function

$$\chi(L)(\alpha) = \begin{cases} 1 & \text{if } \alpha \in L; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

$$c_\alpha = \sum_{\epsilon \in \{0,1\}^d} (-1)^{|\epsilon|} \chi(L)(\alpha + \epsilon), \qquad \alpha \in L.$$

Example: interpolation of total degree

For $m \geq 0$, let

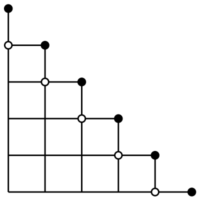
$$L = \{\alpha \in \mathbb{N}_0^d : |\alpha| \leq m\}.$$

The set of maximal points is

$$V = \{\alpha \in \mathbb{N}_0^d : |\alpha| = m\},$$

and L is the union of the blocks B_α with $|\alpha| = m$.
If $d = 2$, the staircase formula gives

$$p(L) = \sum_{|\alpha|=m} p(B_\alpha) - \sum_{|\alpha|=m-1} p(B_\alpha).$$



For $d = 3$, the formula for c_α leads to

$$p(L) = \sum_{|\alpha|=m} p(B_\alpha) - 2 \sum_{|\alpha|=m-1} p(B_\alpha) + \sum_{|\alpha|=m-2} p(B_\alpha).$$