High-order Vector Decomposition with Radial Basis Functions

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Introduction

Helmholtz-Hodge Decomposition:

- $\mathbf{f} \in L_2(\mathbb{R}^d)$:

$$\mathbf{f} = \mathbf{w} + \nabla p, \quad \nabla \cdot \mathbf{w} = 0, \quad \mathbf{w} \perp_{L_2} \nabla p$$

$$\mathbf{f} = \mathbb{P}_{div}\mathbf{f} + \mathbb{P}_{curl}\mathbf{f}$$

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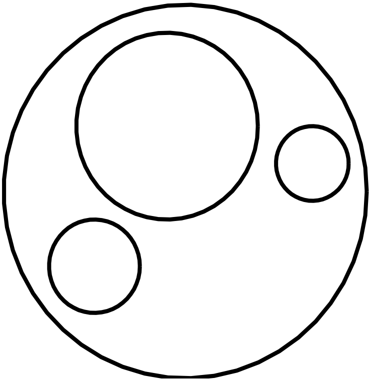

Domain Ω, Boundary Γ

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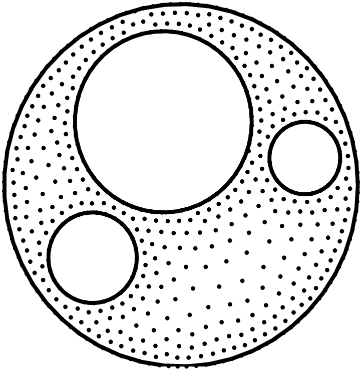

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Nodes $X \subset \Omega$

Sampled Target Field $\mathbf{f}|_X$

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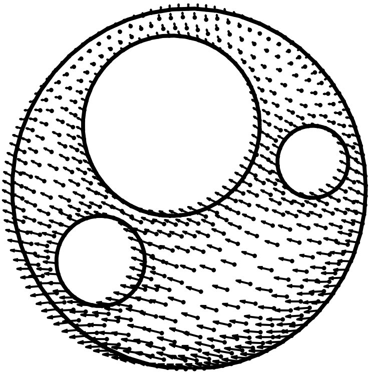

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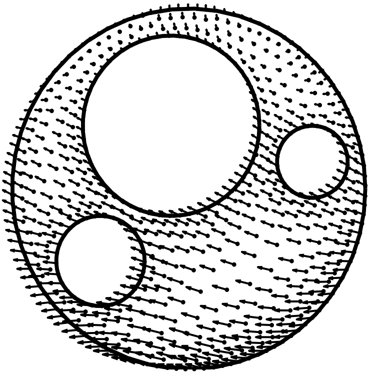
- Given $\mathbf{f}|_X$, find $\mathbf{w} =: \mathbb{P}_L \mathbf{f}$

(Leray projection)



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Customized Matrix-Valued Kernels

Let $\phi(\|\mathbf{x}\|)$ be a scalar-valued RBF. (1990s Amodei, M.N. Benbourhim. Handscomb. Narcowich, Ward.)
(2000s Beatson. Bouhamidi. Dodu, Rabut. Lowitsch. ...Fuselier, Wright. Schraeder, Wendland.)

Matrix Kernel Ψ	Columns of Ψ

- $\Phi_{div} := (-\Delta I + \nabla\nabla^T)\phi$

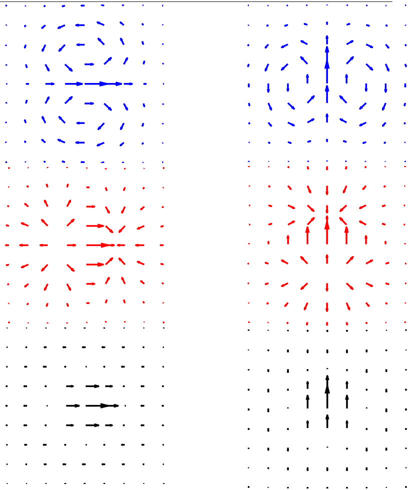
$$\Phi_{div} = \begin{bmatrix} -\phi_{xx} & \phi_{xy} \\ \phi_{xy} & -\phi_{yy} \end{bmatrix}$$

- $\Phi_{curl} := -\nabla\nabla^T\phi$

$$\Phi_{curl} = \begin{bmatrix} -\phi_{yy} & -\phi_{xy} \\ -\phi_{xy} & -\phi_{xx} \end{bmatrix}$$

- $\Phi := -\Delta\phi I$

$$\Phi = \begin{bmatrix} -\Delta\phi & 0 \\ 0 & -\Delta\phi \end{bmatrix}$$



Decomposition Using Customized Kernels

Given $\Psi = \Phi, \Phi_{div}$, or Φ_{curl} and a node set $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{R}^d$
Columns of $\{\Psi(\cdot - \mathbf{x}_j) \,|\, \mathbf{x}_j \in X\}$ form a basis:

$$I_X \mathbf{f} = \sum_{\mathbf{x}_j \in X} \sum_{i=1}^{d} (\Psi(\cdot - \mathbf{x}_j)\mathbf{e}_i)\, c_{ij} = \sum_{\mathbf{x}_j \in X} \Psi(\cdot - \mathbf{x}_j)\mathbf{c}_j, \quad I_X \mathbf{f}|_X = \mathbf{f}|_X \qquad (1)$$

ϕ positive definite $\Rightarrow$ $\Psi = \Phi$, Φ_{div} and Φ_{curl} are all positive definite — (1) is always uniquely solvable.

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Vector Decomposition on $\mathbb{R}^d$: $\mathbf{f} = \mathbb{P}_{div}\mathbf{f} + \mathbb{P}_{curl}\mathbf{f} \qquad (\Phi = \Phi_{div} + \Phi_{curl})$

Proposition: $\mathbb{P}_{div}\Phi(\cdot - x)\mathbf{c} = \Phi_{div}(\cdot - x)\mathbf{c}, \qquad \mathbb{P}_{curl}\Phi(\cdot - x)\mathbf{c} = \Phi_{curl}(\cdot - x)\mathbf{c}.$

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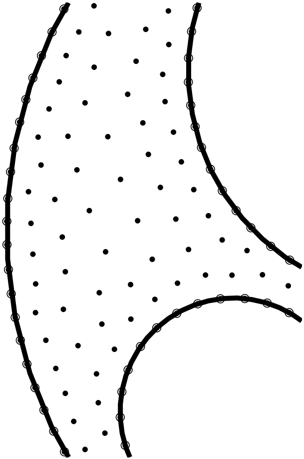
$$\begin{aligned} I_X \mathbf{f} &= \sum_{\mathbf{x}_j \in X} \Phi(\cdot - \mathbf{x}_j)\mathbf{c}_j \\ &= \underbrace{\sum_{\mathbf{x}_j \in X} \Phi_{div}(\cdot - \mathbf{x}_j)\mathbf{c}_j}_{\mathbb{P}_{div} I_X \mathbf{f}} + \underbrace{\sum_{\mathbf{x}_j \in X} \Phi_{curl}(\cdot - \mathbf{x}_j)\mathbf{c}_j}_{\mathbb{P}_{curl} I_X \mathbf{f}} \end{aligned}$$

Studied by Amodei, Benbourhim (1991), Dodu, Rabut (2002, 2004), when ϕ = polyharmonic spline

Custom Interpolant

Given: $X \subset \Omega$, $Y \subset \Gamma$, and $\mathbf{f}|_X$. **Goal**: RBF approximation $\mathbf{s_f}$ such that $\mathbb{P}_{div}\mathbf{s_f} \sim \mathbb{P}_L\mathbf{f}$

Choose Φ as our RBF kernel, search for **generalized interpolant** from $\mathcal{N}_\Phi(\Omega)$.



Requirements	Functionals	Riesz Representers
$\mathbf{s_f}(\mathbf{x}_j) = \mathbf{f}(\mathbf{x}_j)$	$\mathbf{e}_i^T \delta_{\mathbf{x}_j}$	$\Phi(\cdot - \mathbf{x}_j)\mathbf{e}_i$

$$\mathbf{s_f} = \qquad\qquad +$$

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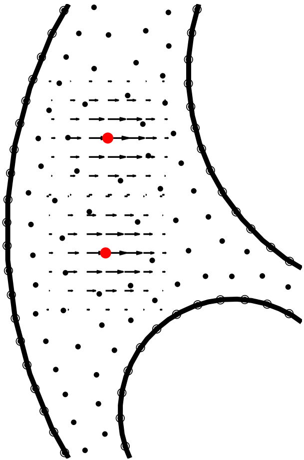

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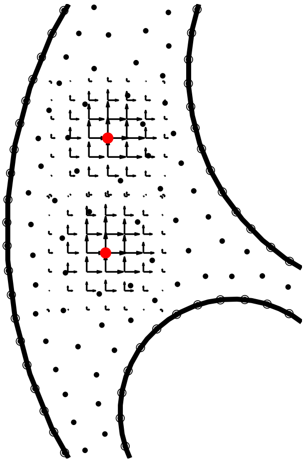

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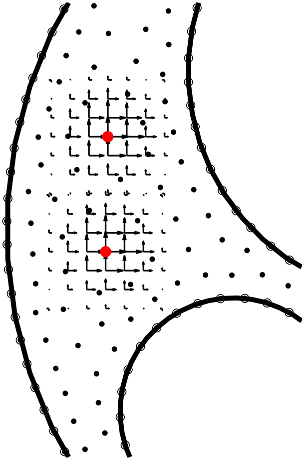

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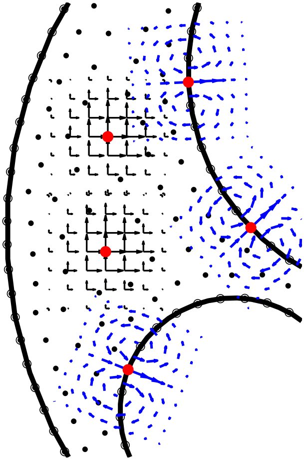

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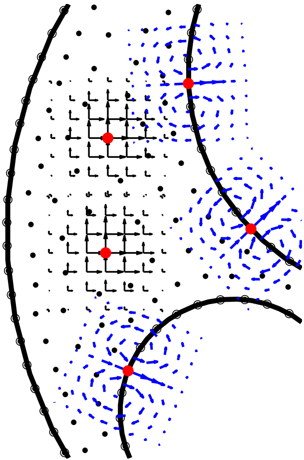

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$$\begin{aligned} \mathbf{s_f}(\mathbf{x}_k) &= \sum_{\mathbf{x}_j \in X} \Phi(\mathbf{x}_k - \mathbf{x}_j)\mathbf{c}_j + \sum_{\mathbf{y}_j \in Y} \Phi_{div}(\mathbf{x}_k - \mathbf{y}_j)\mathbf{n}_j \alpha_j = \mathbf{f}(\mathbf{x}_k) \\ \mathbf{n}_k^T \mathbb{P}_{div} \mathbf{s_f}(\mathbf{y}_k) &= \sum_{\mathbf{x}_j \in X} \mathbf{n}_k^T \Phi_{div}(\mathbf{y}_k - \mathbf{x}_j)\mathbf{c}_j + \sum_{\mathbf{y}_j \in Y} \mathbf{n}_k^T \Phi_{div}(\mathbf{y}_k - \mathbf{y}_j)\mathbf{n}_j \alpha_j = 0 \end{aligned}$$

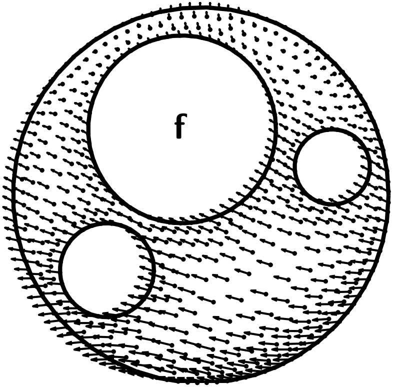
$$\Longrightarrow \underbrace{\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}}_{A_{div}} \begin{bmatrix} \mathbf{c} \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{f}|_X \\ \mathbf{0} \end{bmatrix} \qquad (2)$$

Theorem

If the RBF generator ϕ is positive definite, then for any $X \subset \mathbb{R}^d$, $Y \subset \mathbb{R}^d$, A_{div} is symmetric and positive definite — so (2) is always invertible.

Decomposition on Ω

$$\begin{aligned} \mathbf{s_f} &= \sum_{\mathbf{x}_j \in X} \Phi(\cdot - \mathbf{x}_j)\mathbf{c}_j + \sum_{\mathbf{y}_j \in Y} \Phi_{div}(\cdot - \mathbf{y}_j)\mathbf{n}_j \alpha_j \\ &= \sum_{\mathbf{x}_j \in X} \Phi_{curl}(\cdot - \mathbf{x}_j)\mathbf{c}_j + \sum_{\mathbf{x}_j \in X} \Phi_{div}(\cdot - \mathbf{x}_j)\mathbf{c}_j + \sum_{\mathbf{y}_j \in Y} \Phi_{div}(\cdot - \mathbf{y}_j)\mathbf{n}_j \alpha_j \end{aligned}$$



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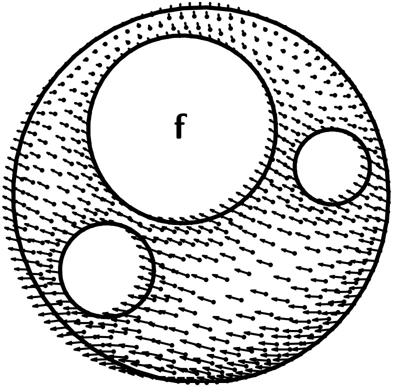
$$= \underbrace{\sum_{\mathbf{x}_j \in X} \Phi_{curl}(\cdot - \mathbf{x}_j)\mathbf{c}_j}_{\mathbb{P}_{curl}\mathbf{s_f}} + \underbrace{\sum_{\mathbf{x}_j \in X} \Phi_{div}(\cdot - \mathbf{x}_j)\mathbf{c}_j + \sum_{\mathbf{y}_j \in Y} \Phi_{div}(\cdot - \mathbf{y}_j)\mathbf{n}_j \alpha_j}_{\mathbb{P}_{div}\mathbf{s_f}}$$



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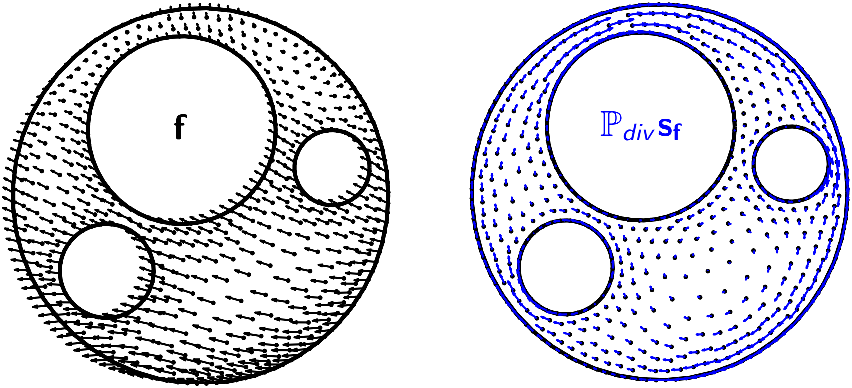

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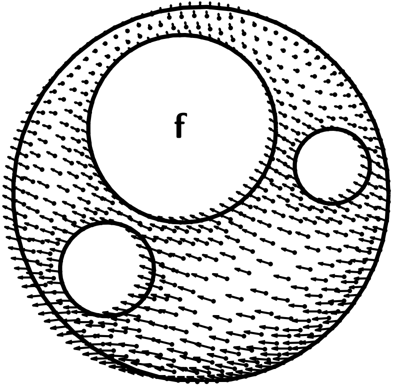

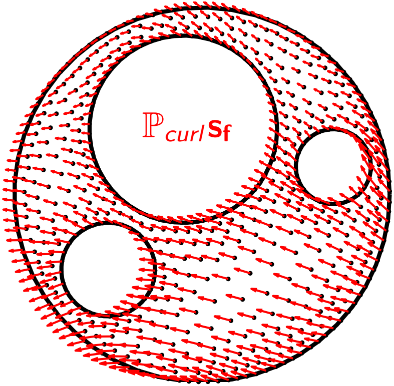

Subtle Nuisances

To find $\mathbf{s_f}$ we solve

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{f}|_X \\ \mathbf{0} \end{bmatrix} \quad (3)$$

Let λ_j be a "boundary functional." We require:

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Lemma

Let Ω be "admissible." Then there is a continuous extension operator $E : \mathbf{H}^\tau(\Omega) \to \widetilde{\mathbf{H}}^\tau(\mathbb{R}^d)$ such that for all $\mathbf{f} \in \mathbf{H}^\tau(\Omega)$ we have $\mathbb{P}_{div} E\mathbf{f}|_\Omega = \mathbb{P}_L \mathbf{f}$.

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$$\lambda_j = \mathbf{n}_j^T \delta_{\mathbf{y}_j} \circ \mathbb{P}_{div}$$

$$\lambda_j(\mathbf{f}) := \lambda_j(E\mathbf{f}) = \mathbf{n}_j^T \mathbb{P}_{div} E\mathbf{f}(\mathbf{y}_j) = \mathbf{n}_j^T \mathbb{P}_L E\mathbf{f}(\mathbf{y}_j) = 0$$

Subtle Nuisances

To find $\mathbf{s_f}$ we solve

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{f}|_X \\ \mathbf{0} \end{bmatrix} \qquad (3)$$

Let λ_j be a "boundary functional." We require:

$$\lambda_j(\mathbf{f}) = \lambda_j(\mathbf{s_f}) \quad \Rightarrow \quad \mathbf{n}_j^T \mathbb{P}_{div} \mathbf{s_f}(\mathbf{y}_j) = 0 = \mathbf{n}_j^T \mathbb{P}_L \mathbf{f}(\mathbf{y}_j) \text{ !?}$$

Lemma

Let Ω be "admissible." Then there is a continuous extension operator $E : \mathbf{H}^\tau(\Omega) \to \widetilde{\mathbf{H}}^\tau(\mathbb{R}^d)$ such that for all $\mathbf{f} \in \mathbf{H}^\tau(\Omega)$ we have $\mathbb{P}_{div} E\mathbf{f}|_\Omega = \mathbb{P}_L \mathbf{f}$.

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$$\lambda_j(\mathbf{f}) := \lambda_j(E\mathbf{f}) = \mathbf{n}_j^T \mathbb{P}_{div} E\mathbf{f}(\mathbf{y}_j) = \mathbf{n}_j^T \mathbb{P}_L E\mathbf{f}(\mathbf{y}_j) = 0$$

Requirements:

- Existence of potential: $\mathbb{P}_L \mathbf{f} = \mathbf{curl}(\psi)$ (holds for multiply-connected Ω...)
- Regularity: If $\mathbf{f} \in \mathbf{H}^\tau(\Omega)$, then $\psi \in \mathbf{H}^{\tau+1}(\Omega)$. (Γ must be smooth enough)

Convergence Rates

$$h_\Omega = \sup_{\mathbf{x}\in\Omega} \min_{\mathbf{x}_j\in X} \|\mathbf{x}-\mathbf{x}_j\|, \quad h_\Gamma = \sup_{\mathbf{x}\in\Gamma} \min_{\mathbf{y}_j\in Y} d(\mathbf{x},\mathbf{y}_j)$$

Lemma (Zeros Lemma [Narcowich, Ward and Wendland])

Let $X \subset \Omega$. For any $\mathbf{u} \in \mathbf{H}^\tau(\Omega)$ such that $\mathbf{u}|_X = 0$, we have

$$\|\mathbf{u}\|_{\mathbf{L}_2(\Omega)} \le C h_\Omega^\tau \|\mathbf{u}\|_{\mathbf{H}^\tau(\Omega)} \quad \left(\|\mathbf{u}\|_{\mathbf{L}_2(\Gamma)} \le C h_\Gamma^{\tau-1/2} \|\mathbf{u}\|_{\mathbf{H}^{\tau-1/2}(\Gamma)}\right).$$

Convergence Rates

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Assume ϕ has finite smoothness: $\widehat{\phi}(\omega) \sim (1+|\omega|^2)^{-\tau-1} \quad (1)$.

Lemma

If the RBF ϕ satisfies (1), then $\mathcal{N}_\Phi(\Omega) = \mathbf{H}^\tau(\Omega)$ with equivalent norms.

Convergence Rates

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Lemma

If the RBF ϕ satisfies (1), then $\mathcal{N}_\Phi(\Omega) = \mathbf{H}^\tau(\Omega)$ with equivalent norms.

Theorem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be compact, smooth and "admissible," and let $X \subset \Omega$, $Y \subset \Gamma$. For any $\mathbf{f} \in \mathbf{H}^\tau(\Omega)$ we have

$$\|\mathbb{P}_L \mathbf{f} - \mathbb{P}_{div} \mathbf{s}_\mathbf{f}\|_{\mathbf{L}_2(\Omega)} \le C \left(h_\Omega^\tau + h_\Gamma^{\tau-1/2} \right) \|\mathbf{f}\|_{\mathbf{H}^\tau(\Omega)}$$

Convergence Rates

$$h_\Omega = \sup_{\mathbf{x}\in\Omega} \min_{\mathbf{x}_j\in X} \|\mathbf{x}-\mathbf{x}_j\|, \quad h_\Gamma = \sup_{\mathbf{x}\in\Gamma} \min_{\mathbf{y}_j\in Y} d(\mathbf{x},\mathbf{y}_j)$$

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$$\|\mathbb{P}_L \mathbf{f} - \mathbb{P}_{div} \mathbf{s}_\mathbf{f}\|_{\mathbf{L}_2(\Omega)} \le C\,(h_\Omega^\tau + h_\Gamma^\tau)\,\|\mathbf{f}\|_{\mathbf{H}^\tau(\Omega)}$$

Smooth Kernels

Let ϕ be such that $\mathcal{N}_\phi(\mathbb{R}^d) \subset H^\tau(\mathbb{R}^d)$ for all $\tau > 0$.

Given $\mathbf{f} = \mathbb{P}_L\mathbf{f} + \nabla g$, we must now assume that

- $\mathbb{P}_L\mathbf{f} = \mathbf{curl}(\psi)$
- ψ, $g \in \mathcal{N}_\phi(\Omega)$.

Theorem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be compact with smooth boundary, and let $X \subset \Omega$, $Y \subset \Gamma$. Fix any $\tau > 0$. Then for any $\mathbf{f}$ as above we have

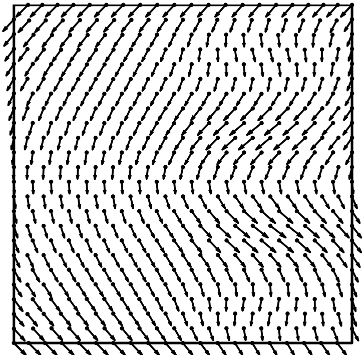
$$\|\mathbb{P}_L\mathbf{f} - \mathbb{P}_{div}\mathbf{s_f}\|_{\mathbf{L}_2(\Omega)} \leq C_\tau \left(h_\Omega^\tau + h_\Gamma^\tau\right) \|\mathbf{f}\|_{\mathcal{N}_\Phi(\Omega)}$$

Convergence on the Square

RBF Generator:

$\phi(r) = e^{-r}(r^5 + 15r^4 + 105r^3 + 420r^2 + 945r + 945)$ C^{10} Matérn Kernel, $\epsilon = 5$

Target: $\mathbf{f} = \begin{bmatrix} -y + \sin(6(y-0.5))e^{-3(x-0.5)^2} \\ -1 \end{bmatrix}$ $\mathcal{N}_\Phi(\Omega) = \mathbf{H}^{5.5}(\Omega)$



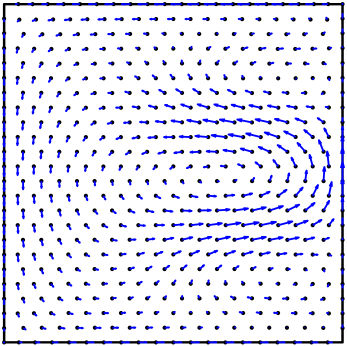
j	N	h	$\lvert \mathbb{P}_{div}\left(\mathbf{s}_\mathbf{f}^{(j)} - \mathbf{s}_\mathbf{f}^{(j+1)}\right) \rvert_\infty$
1	490	0.073	
2	1066	0.048	
3	1902	0.036	
4	2927	0.029	
5	5709	0.022	
6	9410	0.016	—

Convergence on the Square

RBF Generator:

$$\phi(r) = e^{-r}(r^5 + 15r^4 + 105r^3 + 420r^2 + 945r + 945) \quad C^{10} \text{ Matérn Kernel}, \epsilon = 5$$

$$\text{Target: } \mathbf{f} = \begin{bmatrix} -y + \sin(6(y-0.5))e^{-3(x-0.5)^2} \\ -1 \end{bmatrix} \qquad \mathcal{N}_\Phi(\Omega) = \mathbf{H}^{5.5}(\Omega)$$



j	N	h	$\lvert \mathbb{P}_{div}\left(\mathbf{s_f}^{(j)} - \mathbf{s_f}^{(j+1)}\right)\rvert_\infty$
1	490	0.073	0.02301
2	1066	0.048	0.01062
3	1902	0.036	0.00677
4	2927	0.029	0.00719
5	5709	0.022	0.00296
6	9410	0.016	—

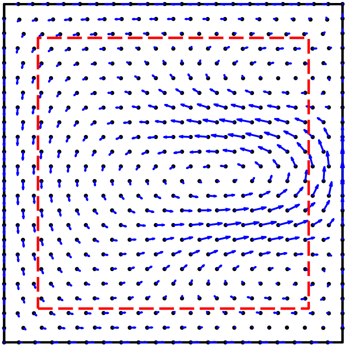
Experimental Error $= \mathcal{O}(h^{1.574})$

Convergence on the Square

RBF Generator:

$$\phi(r) = e^{-r}(r^5 + 15r^4 + 105r^3 + 420r^2 + 945r + 945) \quad C^{10} \text{ Matérn Kernel, } \epsilon = 5$$

$$\text{Target: } \mathbf{f} = \begin{bmatrix} -y + \sin(6(y-0.5))e^{-3(x-0.5)^2} \\ -1 \end{bmatrix} \qquad \mathcal{N}_\Phi(\Omega) = \mathbf{H}^{5.5}(\Omega)$$



j	N	h	$\lvert \mathbb{P}_{div}\left(\mathbf{s}_\mathbf{f}^{(j)} - \mathbf{s}_\mathbf{f}^{(j+1)}\right)\rvert_\infty$
1	490	0.073	0.00209
2	1066	0.048	0.00025
3	1902	0.036	0.00019
4	2927	0.029	0.00008
5	5709	0.022	0.00006
6	9410	0.016	—

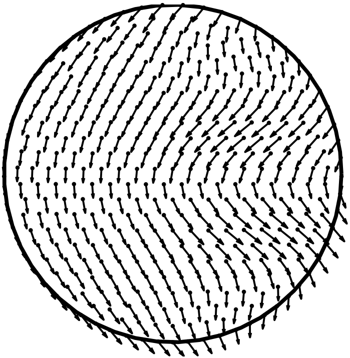
Experimental Error $= \mathcal{O}(h^{2.3})$

Convergence on the Disk

RBF Generator:

$$\phi(r) = e^{-r}(r^5 + 15r^4 + 105r^3 + 420r^2 + 945r + 945) \quad C^{10} \text{ Matérn Kernel, } \epsilon = 5$$

$$\text{Target: } \mathbf{f} = \begin{bmatrix} -y + \sin(6(y-0.5))e^{-3(x-0.5)^2} \\ -1 \end{bmatrix} \qquad \mathcal{N}_\Phi(\Omega) = \mathbf{H}^{5.5}(\Omega)$$



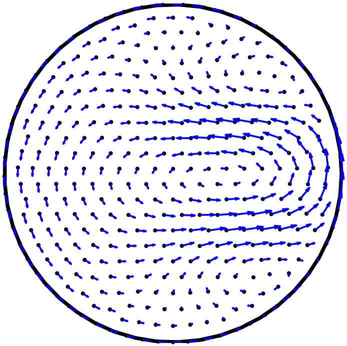
j	N	h	$\lvert\mathbb{P}_{div}\left(\mathbf{s}_\mathbf{f}^{(j)} - \mathbf{s}_\mathbf{f}^{(j+1)}\right)\rvert_\infty$
1	362	0.075	
2	808	0.050	
3	1452	0.037	
4	2267	0.030	
5	4441	0.021	
6	7330	0.016	—

Convergence on the Disk

RBF Generator:

$$\phi(r) = e^{-r}(r^5 + 15r^4 + 105r^3 + 420r^2 + 945r + 945) \quad C^{10} \text{ Matérn Kernel, } \epsilon = 5$$

$$\text{Target: } \mathbf{f} = \begin{bmatrix} -y + \sin(6(y-0.5))e^{-3(x-0.5)^2} \\ -1 \end{bmatrix} \qquad \mathcal{N}_\Phi(\Omega) = \mathbf{H}^{5.5}(\Omega)$$



j	N	h	$\lvert \mathbb{P}_{div}\left(\mathbf{s}_\mathbf{f}^{(j)} - \mathbf{s}_\mathbf{f}^{(j+1)}\right) \rvert_\infty$
1	362	0.075	0.0010432
2	808	0.050	0.0001678
3	1452	0.037	0.0000383
4	2267	0.030	0.0000120
5	4441	0.021	0.0000019
6	7330	0.016	—

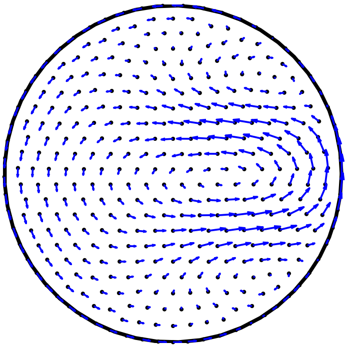
Experimental Error $= \mathcal{O}(h^{5.36})$

Convergence on the Disk

RBF Generator:

$\phi(r) = \sqrt{1 + r^2}$ C^∞ Multiquadric, $\epsilon = 1.5$

Target: $\mathbf{f} = \begin{bmatrix} -y + \sin(6(y-0.5))e^{-3(x-0.5)^2} \\ -1 \end{bmatrix}$ $\mathcal{N}_\Phi(\Omega) = ??$



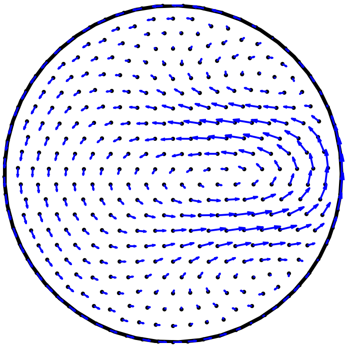
j	N	h	$\lvert \mathbb{P}_{div} \left(\mathbf{s}_\mathbf{f}^{(j)} - \mathbf{s}_\mathbf{f}^{(j+1)} \right) \rvert_\infty$
1	362	0.075	0.0025466
2	808	0.050	0.0002744
3	1452	0.037	0.0000193
4	2267	0.030	0.0000053
5	4441	0.021	0.0000086
6	7330	0.016	—

Convergence on the Disk

RBF Generator:

$\phi(r) = \sqrt{1 + r^2}$ C^∞ Multiquadric, $\epsilon = 1.5$

Target: $\mathbf{f} = \begin{bmatrix} -y + \sin(6(y-0.5))e^{-3(x-0.5)^2} \\ -1 \end{bmatrix}$ $\mathcal{N}_\Phi(\Omega) = ??$



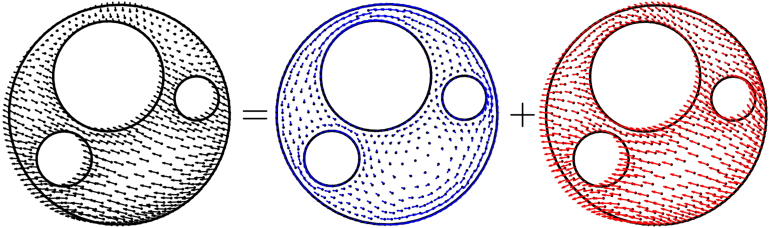
j	N	h	$\lvert \mathbb{P}_{div}\left(\mathbf{s}_\mathbf{f}^{(j)} - \mathbf{s}_\mathbf{f}^{(j+1)}\right)\rvert_\infty$
1	362	0.075	0.0025466
2	808	0.050	0.0002744
3	1452	0.037	0.0000193
4	2267	0.030	0.0000053
5	4441	0.021	0.0000086
6	7330	0.016	—

Experimental Error $= \mathcal{O}(h^{6.85})$

(excluding j=5)

Remarks...

- Efficiency, conditioning, etc.
- Other decompositions/boundary conditions possible:



Remarks...

- Efficiency, conditioning, etc.
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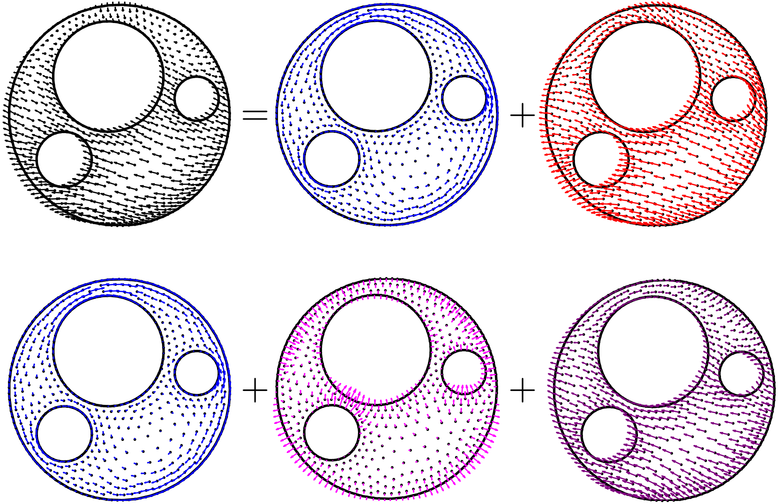

Thanks!