The Method of Fundamental Solutions in Solving Coupled Boundary Value Problems for EEG/MEG

Salvatore Ganci

salvatore.ganci@unipa.it
DEIM, University of Palermo, Italy

Joint work with **G. Ala**, **G. Fasshauer**, **E. Francomano** and **M. McCourt**

MAIA 2013
Erice, 25-30 September, 2013

Outline

Background

- **What are EEG and MEG?**
  EEG and MEG are two electromagnetic techniques for brain activity investigation, i.e. to locate active neural sources
- **How they work?**
  Neural sources (location and amplitude) are reconstructed starting from measurements of electric potential on the scalp (EEG) or magnetic field near the head (MEG). This is a typical *inverse problem*.
- **What is needed to perform them?**
  1. A data set (measurements)
  2. An inverse algorithm
  3. An *efficient* and *accurate* forward solver

Model for the Head

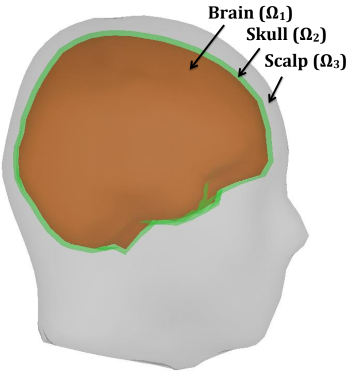


Figure 1 : Compartment model for the head

- The head can be modeled as a **linear**, **piecewise homogeneous**, volume conductor domain $\Omega \subset \mathbb{R}^3$ formed by L **nested layers**.
  Let $\boldsymbol{p}$ be a point in Ω.
- A model with three layers ($L = 3$) is common: *brain*, *skull* and *scalp*.
- Let Ω_ℓ and $\partial\Omega_\ell$ be the ℓ-th layer in the domain Ω, with known conductivity σ_ℓ, and its boundary, respectively.
- The medium surrounding the head is the air and it can be considered as an unbounded region of null electrical conductivity.

Electromagnetic Modeling of the Brain Activity

The **forward problem for the electric potential** $\phi(\boldsymbol{p})$ can be formulated as the following BVP:

$$\begin{cases} \sigma_\ell(\boldsymbol{p})\nabla^2\phi(\boldsymbol{p}) = S_\ell(\boldsymbol{p}), & \boldsymbol{p} \in \Omega_\ell \\ \phi(\boldsymbol{p}^-) = \phi(\boldsymbol{p}^+), & \boldsymbol{p} \in \partial\Omega_\ell \cap \partial\Omega_{\ell+1} \\ \sigma_\ell \boldsymbol{n}(\boldsymbol{p}) \cdot \nabla\phi(\boldsymbol{p}^-) = \sigma_{\ell+1}\boldsymbol{n}(\boldsymbol{p}) \cdot \nabla\phi(\boldsymbol{p}^+), & \boldsymbol{p} \in \partial\Omega_\ell \cap \partial\Omega_{\ell+1} \end{cases}$$

where:

- $S_\ell(\boldsymbol{p}) = \begin{cases} \nabla \cdot (\mathbf{Q}\delta(\boldsymbol{p} - \boldsymbol{p}')) & \text{neural source in } \boldsymbol{p}' \in \Omega_\ell \\ 0 & \text{otherwise} \end{cases}$
- $\boldsymbol{n}(\boldsymbol{p})$ is the outward unit vector normal to the interface $\partial\Omega_\ell \cap \partial\Omega_{\ell+1}$ at $\boldsymbol{p}$
- $\boldsymbol{p}^-$ and $\boldsymbol{p}^+$ are limit points for two spatial sequences converging to $\boldsymbol{p}$ from inside and from outside the interface, respectively

Electromagnetic Modeling of the Brain Activity

The **forward problem for the magnetic field** can be formulated starting from the forward problem for the electric potential.

In fact, the Maxwell's equations yield:

$$\nabla^2 \mathbf{B}(\boldsymbol{p}) = -\mu \nabla \times \mathbf{J}(\boldsymbol{p}) \tag{1}$$

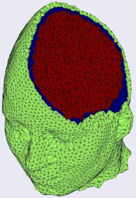
where $\mathbf{B}(\boldsymbol{p})$ is the magnetic induction, μ is the permeability of the medium and the current density $\mathbf{J}(\boldsymbol{p})$ is known once $\phi(\boldsymbol{p})$ is known.

The solution of (1) with condition of null magnetic field at infinite distance from sources, is known as **Ampère-Laplace law** [Sarvas (1987)]:

$$\mathbf{B}(\boldsymbol{p}) = -\frac{\mu}{4\pi} \int_{\Omega} \sigma(\boldsymbol{p}^*) \nabla \phi(\boldsymbol{p}^*) \times \frac{\boldsymbol{p} - \boldsymbol{p}^*}{\|\boldsymbol{p} - \boldsymbol{p}^*\|^3} \mathrm{d}\Omega^*$$

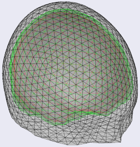
State of the Art

Finite Element Method



- Domain method → **3D meshes**
- Very costly

Boundary Element Method



- Boundary method → **2D meshes**
- **Comparable to FEM in accuracy** [Adde et al. (2003)]
- **Implemented in popular toolboxes for EEG/MEG analysis**, e.g. FieldTrip [Oostenveld et al. (2011)], Brainstorm [Tadel et al. (2011)].

Motivation

Drawbacks of the state of the art solvers:

1. High quality meshes are needed to avoid **mesh-related artifacts** in reconstructed neural activation patterns
2. **Mesh generation is a complex and time consuming pre-processing task**, even with automatic algorithms
3. **Numerical integration is required** and turns out to be the dominating computational task in the process
4. **Complex computer codes** (not flexible).

What could be done to overcome these difficulties?

Motivation

The Method of Particular Solutions (MPS) allows for the application of the **Method of Fundamental Solutions (MFS)**

1. **Boundary-type method**, like BEM
2. **No meshing** is required: ability to handle complex geometries in an easy way
3. **No numerical integration** is required
4. **Accuracy**: potential for exponential convergence with smooth data and domains [Cheng (1987); Katsurada (1994); Katsurada and Okamoto (1996)]
5. **Flexibility**: easy implementation

The underlying idea

The MFS is a kernel-based method, introduced during 60's [Kupradze and Aleksidze (1964a,b); Kupradze (1967)]
Let's consider a homogeneous elliptic PDE of the form:

$$\mathcal{L}u(\boldsymbol{p}) = 0, \quad \boldsymbol{p} \in \Omega \subseteq \mathbb{R}^3 \tag{2}$$

Like BEM, MFS is applicable when a *fundamental solution* of the PDE is known.

Definition – Fundamental solution

A fundamental solution of the PDE (2) a function $K(\boldsymbol{p}, \boldsymbol{q})$ such that
$$\mathcal{L}K(\boldsymbol{p}, \boldsymbol{q}) = -\delta(\boldsymbol{p} - \boldsymbol{q}), \quad \boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^3$$

$\boldsymbol{q}$ is called the *singularity point* (or *source point*) of the fundamental solution since K is defined everywhere except there, where it is singular.

The underlying idea

The idea of the MFS is to estimate the solution by means of a linear combination of fundamental solutions of the governing PDE:

$$u(\boldsymbol{p}) \approx \hat{u}(\boldsymbol{p}) = \sum_{j=1}^{\#\Xi} c_j K(\boldsymbol{p}, \boldsymbol{\xi}_j), \quad \boldsymbol{p} \in \Omega, \quad \boldsymbol{\xi}_j \in \Xi \tag{3}$$

were Ξ is a set of *source points* placed on a **fictitious boundary outside the physical domain**.

The coefficients c_j have to be determined by imposing (3) to satisfy the boundary conditions:

$$\mathcal{T}u(\boldsymbol{p}) = f^{\partial\Omega}(\boldsymbol{p}), \quad \boldsymbol{p} \in \partial\Omega$$

at a set P of *collocation points*:

$$\sum_{j=1}^{\#\Xi} c_j \mathcal{T}K(\boldsymbol{p}_i, \boldsymbol{\xi}_j) = f^{\partial\Omega}(\boldsymbol{p}_i) \quad \boldsymbol{p}_i \in P, \quad \boldsymbol{\xi}_j \in \Xi \tag{4}$$

Inhomogeneous problems

Let's consider an **inhomogeneous** BVP of the form:

$$\begin{cases} \mathcal{L}u(\boldsymbol{p}) = f^{\Omega}(\boldsymbol{p}), & \boldsymbol{p} \in \Omega \subseteq \mathbb{R}^3 \\ \mathcal{T}u(\boldsymbol{p}) = f^{\partial\Omega}(\boldsymbol{p}), & \boldsymbol{p} \in \partial\Omega \end{cases} \tag{5}$$

It can be reduced to a homogeneous problem by the **Method of Particular Solutions** (**MPS**), i.e. by splitting u into a *particular solution* u_p and its associated homogeneous solution u_h:

$$u = u_h + u_p$$

Definition – Particular solution

A particular solution of the BVP (5) is a function u_p on $\Omega \cup \partial\Omega$ which satisfies the inhomogeneous PDE but not necessarily the boundary conditions.

Then we get the homogenous BVP:

$$\begin{cases} \mathcal{L}u_h(\boldsymbol{p}) = f^{\Omega}(\boldsymbol{p}) - \mathcal{L}u_p(\boldsymbol{p}) = 0, & \boldsymbol{p} \in \Omega \\ \mathcal{T}u_h(\boldsymbol{p}) = f^{\partial\Omega}(\boldsymbol{p}) - \mathcal{T}u_p(\boldsymbol{p}), & \boldsymbol{p} \in \partial\Omega \end{cases}$$

Application to the EEG potential problem

Let's apply the MFS, via MPS, to the EEG potential problem.

1. The fundamental solution for the Laplace equation in 3D is:

$$K(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{4\pi \|\boldsymbol{p} - \boldsymbol{q}\|}$$

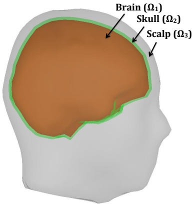
2. An analytical expression for a function $\phi_p(\boldsymbol{p})$ that satisfies the equation

$$\sigma \nabla^2 \phi(\boldsymbol{p}) = \nabla \cdot (\mathbf{Q} \delta(\boldsymbol{p} - \boldsymbol{p}'))$$

in an unbounded domain is known [Sarvas (1987)]:

$$\phi_p(\boldsymbol{p}) = \frac{1}{4\pi\sigma} \frac{\boldsymbol{p} - \boldsymbol{p}'}{\|\boldsymbol{p} - \boldsymbol{p}'\|^3} \cdot \mathbf{Q}$$

Application to the EEG potential problem



The EEG potential problem in Ω can be addressed by considering a number L of **coupled** BVPs interacting through the boundary conditions.
By introducing the parameter

$$\alpha_\ell = \begin{cases} 1, & \text{neural source in } \Omega_\ell \\ 0, & \text{otherwise} \end{cases}$$

the potential function in each layer can be expressed as:

$$\phi_\ell(\boldsymbol{p}) = \phi_{h,\ell}(\boldsymbol{p}) + \alpha_\ell \phi_{p,\ell}(\boldsymbol{p})$$

$\phi_{h,\ell}$ is given by the solution of the following homogeneous BVP:

$$\begin{cases} \nabla^2 \phi_{h,\ell}(\boldsymbol{p}) = 0, & \boldsymbol{p} \in \Omega_\ell \\ \phi_{h,\ell}(\boldsymbol{p}) - \phi_{h,\ell+1}(\boldsymbol{p}) = \alpha_{\ell+1}\phi_{p,\ell+1}(\boldsymbol{p}) - \alpha_\ell \phi_{p,\ell}(\boldsymbol{p}), & \boldsymbol{p} \in \partial\Omega_\ell \cap \partial\Omega_{\ell+1} \\ \sigma_\ell \boldsymbol{n}(\boldsymbol{p}) \cdot \nabla\phi_{h,\ell}(\boldsymbol{p}) - \sigma_{\ell+1}\boldsymbol{n}(\boldsymbol{p}) \cdot \nabla\phi_{h,\ell+1}(\boldsymbol{p}) = & \\ = \alpha_{\ell+1}\sigma_{\ell+1}\boldsymbol{n}(\boldsymbol{p}) \cdot \nabla\phi_{p,\ell+1}(\boldsymbol{p}) - \alpha_\ell \sigma_\ell \boldsymbol{n}(\boldsymbol{p}) \cdot \nabla\phi_{p,\ell}(\boldsymbol{p}) & \boldsymbol{p} \in \partial\Omega_\ell \cap \partial\Omega_{\ell+1} \end{cases}$$

Application to the EEG potential problem

- The homogeneous solution is approximated by:

$$\hat{\phi}_{h,\ell}(\boldsymbol{p}) = \sum_{j=1}^{\#\Xi_\ell} c_j^\ell K(\boldsymbol{p}, \boldsymbol{\xi}_j), \quad \boldsymbol{p} \in \Omega_\ell, \quad \boldsymbol{\xi}_j \in \Xi_\ell \tag{6}$$

  where Ξ_ℓ is the set of source points relative to the layer Ω_ℓ.
- In order to estimate the L sets of coefficients $\{\boldsymbol{c}^\ell\}_{\ell=1}^L$, the collocation has to be performed on each interface.
- Let $P_{\ell,\ell+1}^{\mathcal{D}}$ and $P_{\ell,\ell+1}^{\mathcal{N}}$ be the sets of collocation points on the interface between the layer ℓ and the layer $\ell+1$ where Dirichlet conditions and Neumann conditions, respectively, are intended to be imposed.

Application to the EEG potential problem

The collocation yields the following **coupled linear system**:

$$
\begin{bmatrix}
\mathsf{D}^1_{1,2} & \mathsf{D}^2_{1,2} & & & & & & \\
\mathsf{N}^1_{1,2} & \mathsf{N}^2_{1,2} & & & & & & \\
 & \mathsf{D}^2_{2,3} & \mathsf{D}^3_{2,3} & & & & & \\
 & \mathsf{N}^2_{2,3} & \mathsf{N}^3_{2,3} & & & & & \\
 & & & \ddots & & & & \\
 & & & & \mathsf{D}^{\ell}_{\ell,\ell+1} & \mathsf{D}^{\ell+1}_{\ell,\ell+1} & & \\
 & & & & \mathsf{N}^{\ell}_{\ell,\ell+1} & \mathsf{N}^{\ell+1}_{\ell,\ell+1} & & \\
 & & & & & \ddots & & \\
 & & & & & & \mathsf{D}^{L-1}_{L-1,L} & \mathsf{D}^{L}_{L-1,L} \\
 & & & & & & \mathsf{N}^{L-1}_{L-1,L} & \mathsf{N}^{L}_{L-1,L} \\
 & & & & & & & \mathsf{N}^{L}_{L,L+1}
\end{bmatrix}
\begin{bmatrix}
\boldsymbol{c}^1 \\ \boldsymbol{c}^2 \\ \vdots \\ \boldsymbol{c}^{\ell} \\ \vdots \\ \boldsymbol{c}^{L-1} \\ \boldsymbol{c}^{L}
\end{bmatrix}
=
\begin{bmatrix}
\boldsymbol{b}^{\mathcal{D}}_{1,2} \\ \boldsymbol{b}^{\mathcal{N}}_{1,2} \\ \boldsymbol{b}^{\mathcal{D}}_{2,3} \\ \boldsymbol{b}^{\mathcal{N}}_{2,3} \\ \vdots \\ \boldsymbol{b}^{\mathcal{D}}_{\ell,\ell+1} \\ \boldsymbol{b}^{\mathcal{N}}_{\ell,\ell+1} \\ \vdots \\ \boldsymbol{b}^{\mathcal{D}}_{L-1,L} \\ \boldsymbol{b}^{\mathcal{N}}_{L-1,L} \\ \boldsymbol{b}^{\mathcal{N}}_{L,L+1}
\end{bmatrix}
$$

$$
\begin{aligned}
(\mathsf{D}^{\ell}_{\ell,\ell+1})_{ij} &= K(\boldsymbol{p}_i, \boldsymbol{\xi}_j) & \boldsymbol{p}_i &\in P^{\mathcal{D}}_{\ell,\ell+1}, \quad \boldsymbol{\xi}_j \in \Xi_{\ell} \\
(\mathsf{D}^{\ell+1}_{\ell,\ell+1})_{ij} &= -K(\boldsymbol{p}_i, \boldsymbol{\xi}_j) & \boldsymbol{p}_i &\in P^{\mathcal{D}}_{\ell,\ell+1}, \quad \boldsymbol{\xi}_j \in \Xi_{\ell+1} \\
(\mathsf{N}^{\ell}_{\ell,\ell+1})_{ij} &= \sigma_{\ell} \boldsymbol{n}(\boldsymbol{p}_i) \cdot \nabla K(\boldsymbol{p}_i, \boldsymbol{\xi}_j) & \boldsymbol{p}_i &\in P^{\mathcal{N}}_{\ell,\ell+1}, \quad \boldsymbol{\xi}_j \in \Xi_{\ell} \\
(\mathsf{N}^{\ell+1}_{\ell,\ell+1})_{ij} &= -\sigma_{\ell+1} \boldsymbol{n}(\boldsymbol{p}_i) \cdot \nabla K(\boldsymbol{p}_i, \boldsymbol{\xi}_j) & \boldsymbol{p}_i &\in P^{\mathcal{N}}_{\ell,\ell+1}, \quad \boldsymbol{\xi}_j \in \Xi_{\ell+1} \\
(\boldsymbol{b}^{\mathcal{D}}_{\ell,\ell+1})_i &= \alpha_{\ell+1} \phi_{p,\ell+1}(\boldsymbol{p}_i) - \alpha_{\ell} \phi_{p,\ell}(\boldsymbol{p}_i) & \boldsymbol{p}_i &\in P^{\mathcal{D}}_{\ell,\ell+1} \\
(\boldsymbol{b}^{\mathcal{N}}_{\ell,\ell+1})_i &= \alpha_{\ell+1} \sigma_{\ell+1} \boldsymbol{n}(\boldsymbol{p}_i) \cdot \nabla \phi_{p,\ell+1}(\boldsymbol{p}_i) - \alpha_{\ell} \sigma_{\ell} \boldsymbol{n}(\boldsymbol{p}_i) \cdot \nabla \phi_{p,\ell}(\boldsymbol{p}_i) & \boldsymbol{p}_i &\in P^{\mathcal{N}}_{\ell,\ell+1}
\end{aligned}
$$

Simulation data

Homogeneous sphere

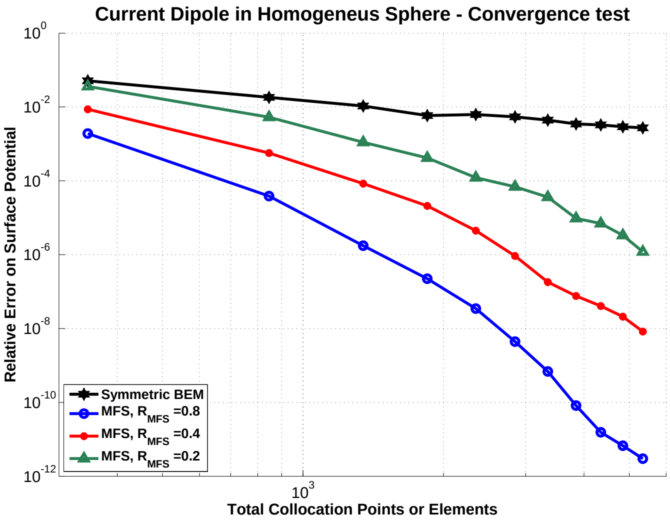
Analytical solution in [Yao (2000)]

Problem data:

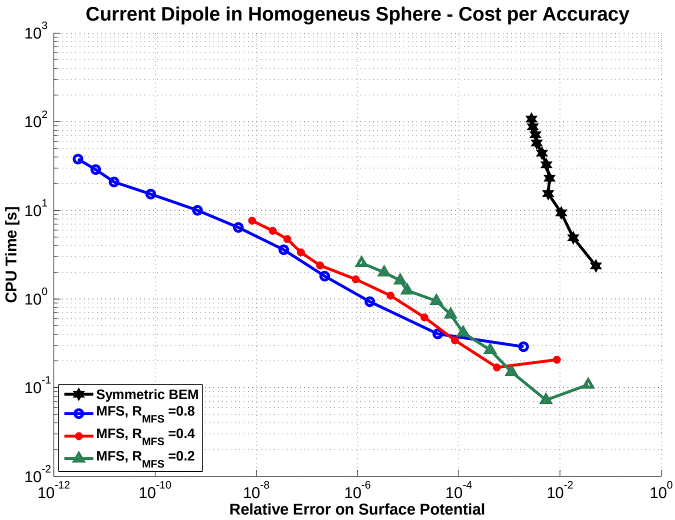
- Sphere radius: 0.1 m
- Conductivity: 0.2 S/m
- Source position: $(0, 0, 0.06)$ [m]
- Source moment: $(1, 0, 0)$ [Am]

Simulations with different ratios R_{MFS} between the no. source points and the number of collocation points

Convergence



Cost vs. Relative Error



Simulation data

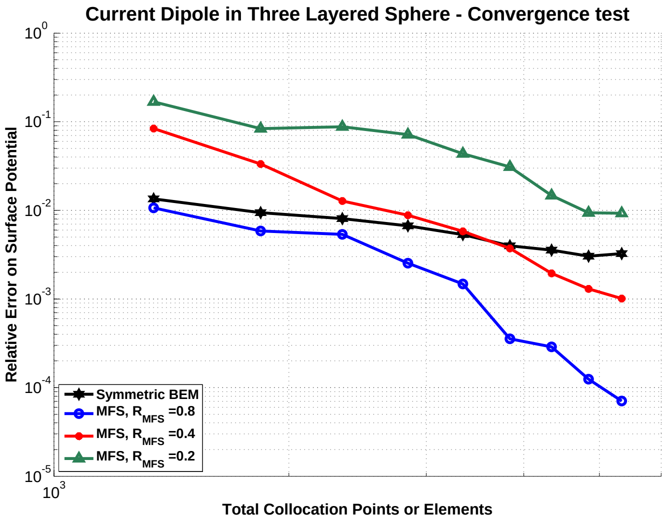
Three layered sphere

Semi-analytical solution in [Zhang (1995)]

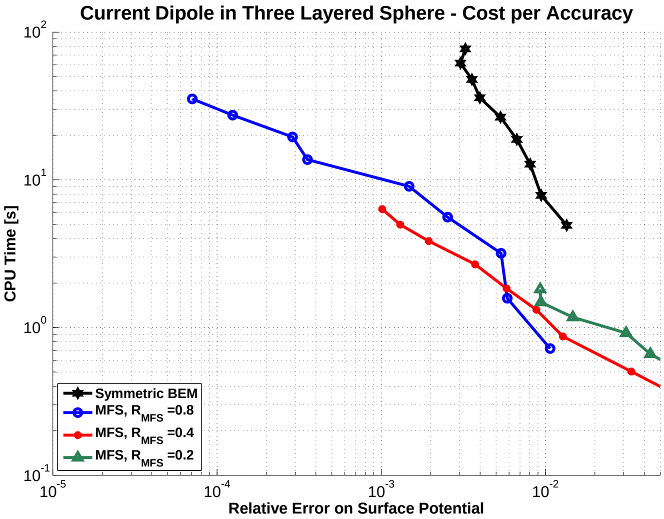
Problem data:

- Sphere radii: $R_1 = 0.087$ m, $R_2 = 0.092$ m, $R_3 = 0.1$ m
- Conductivities: $\sigma_1 = 0.33$ S/m, $\sigma_2 = 0.0125$ S/m, $\sigma_3 = 0.33$ S/m
- Source position: $(0, 0, 0.052)$ [m]
- Source moment: $(1, 0, 0)$ [Am]

Convergence



Cost vs. Relative Error



Conclusions

1. The MFS via MPS has been proposed to address the EEG/MEG forward problem
2. This permits to get rid of complex and time consuming meshing algorithms, mesh related artifacts and troublesome numerical integration
3. The implementation of the presented method is straightforward: unlike BEM solvers, the code is very flexible
4. Simulations results for simplified head geometries showed:
   - very good agreement with (semi)analytic solutions
   - clear superiority with respect to BEM from a cost vs. accuracy standpoint

References I

Adde, G., Clerc, M., Faugeras, O., Keriven, R., Kybic, J., and Papadopoulo, T. (2003). Symmetric bem formulation for the m/eeg forward problem. In Taylor, C. and Noble, J., editors, *Information Processing in Medical Imaging*, volume 2732 of *Lecture Notes in Computer Science*, pages 524–535. Springer Berlin Heidelberg.

Cheng, R. (1987). *Delta-trigonometric and spline methods using the single-layer potential representation*. PhD thesis, University of Maryland.

Katsurada, M. (1994). Charge simulation method using exterior mapping functions. *Japan journal of industrial and applied mathematics*, 11(1):47–61.

Katsurada, M. and Okamoto, H. (1996). The collocation points of the fundamental solution method for the potential problem. *Computers & Mathematics with Applications*, 31(1):123–137.

References II

Kupradze, V. (1967). On the approximate solution of problems in mathematical physics. *Russian Mathematical Surveys*, 22:58–108.

Kupradze, V. and Aleksidze, M. (1964a). A method for the approximate solution of limiting problems in mathematical physics. *U.S.S.R. Computational Mathematics and Mathematical Physics*, 4:199–205.

Kupradze, V. and Aleksidze, M. (1964b). The method of functional equations for the approximate solution of certain boundary value problems. *U.S.S.R. Computational Mathematics and Mathematical Physics*, 4:82–126.

Oostenveld, R., Fries, P., Maris, E., and Schoffelen, J. (2011). Fieldtrip: open source software for advanced analysis of meg, eeg, and invasive electrophysiological data. *Computational intelligence and neuroscience*, 2011:1.

Sarvas, J. (1987). Basic mathematical and electromagnetic concepts of the biomagnetic inverse problem. *Physics in Medicine and Biology*, 32(1):11.

References III

Tadel, F., Baillet, S., Mosher, J., Pantazis, D., and Leahy, R. (2011). Brainstorm: a user-friendly application for meg/eeg analysis. *Computational intelligence and neuroscience*, 2011:8.

Yao, D. (2000). Electric potential produced by a dipole in a homogeneous conducting sphere. *IEEE Transactions on Biomedical Engineering*, 47(7):964–966.

Zhang, Z. (1995). A fast method to compute surface potentials generated by dipoles within multilayer anisotropic spheres. *Physics in Medicine and Biology*, 40(3):335.