New Results Concerning the Gasca-Maeztu Conjecture

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Synopsis

- Introduction and Notation
- The Gasca-Maeztu Conjecture
- Maximal Lines and Maximal Curves
- n-dependent Sets
- Sets Satisfying the Chung-Yao Geometric Condition
- The m-distribution Sequence of a Node
- Lines Used by Several Nodes
- Some Ideas in the Proof for $n = 5$
- Outlook

Joint work with **Hakop Hakopian** and **Georg Zimmermann**

Notation

- Π_n the space of algebraic polynomials of (total) degree at most $n \geq 1$.
- $X \subset \mathbb{R}^2$ with $|X| = \dim \Pi_n = \binom{n+2}{2}$.

General Assumption:

- X is n-*poised* (n-regular, n-correct), i.e.,

$$p \in \Pi_n \quad \text{and} \quad p(\mathbf{x}) = 0\ ,\ \mathbf{x} = (x, y) \in X \quad \Longrightarrow \quad p = 0\ .$$

- Equivalently, for any real (or complex) data $\{c_{\mathbf{x}}\ ,\ \mathbf{x} \in X\}$, there is a unique polynomial $p \in \Pi_n$, interpolating these data:

$$p(\mathbf{x}) = c_{\mathbf{x}}\ , \quad \mathbf{x} \in X\ .$$

- Equivalently, the Vandermonde with respect to X and any basis of Π_n is regular.

Here, in case we use the monomial basis, the Vandermonde $\mathcal{V} = \mathcal{V}_n(X)$ consists of rows

$$[\ \cdots\ \mathbf{x}^\alpha\ \ \cdots\ ]_{\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}_0^2\ ,\alpha_1+\alpha_2\leq n}\ , \quad \text{for} \quad \mathbf{x} = (x_1, x_2) \in X\ , \quad \mathbf{x}^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2}\ .$$

Fundamental Polynomials

n-poisedness is equivalent to the fact that for any $\mathbf{x} \in X$, and the data

$$e_{\mathbf{x}} = (\delta_{\mathbf{x},\mathbf{y}})_{\mathbf{y} \in X} \ ,$$

we have a unique (so-called Lagrange) *fundamental polynomial* $p_{\mathbf{x}} \in \Pi_n$, determined by

$$p_{\mathbf{x}}(\mathbf{y}) = \delta_{\mathbf{x},\mathbf{y}} \ , \quad \mathbf{y} \in X \ , \tag{1}$$

and the polynomial interpolating the data $(c_{\mathbf{x}})_{\mathbf{x} \in X}$ may be written (in its Lagrange form) as

$$p = \sum_{\mathbf{x} \in X} c_{\mathbf{x}} \, p_{\mathbf{x}} \ .$$

Remark. Fundamental polynomials satisfying (1) can be also considered for not necessarily n-poised sets X, in particular if $|X| < \dim \Pi_n$. In this case, for some point $\mathbf{x} \in X$ such a fundamental polynomial may not exist and/or may not be unique.

The Gasca-Maeztu Conjecture

The **Gasca-Maeztu conjecture** (for short: **GM-conjecture** or $\mathrm{GM}_{\mathbf{n}}$**-conjecture**) is based on the assumption that *each fundamental polynomial* $p_{\mathbf{x}}$*,* $\mathbf{x} \in X$*, factors into linear polynomials.*

It states that *under this assumption at least* $n+1$ *nodes from the set* X *must be collinear*.

It was first stated in **Gasca-Maeztu** [2]; see also **Carnicer-Gasca** [3].

Surprisingly, up to now the conjecture is verified only for $n \leq 5$.

$n = 1$ Here, $|X| = \dim \Pi_1 = 3$, and X is 1-poised iff the three nodes are not collinear.

For each node $\mathbf{x} \in X$, the fundamental polynomial $p_{\mathbf{x}}$ is (the linear polynomial whose zero set is) the line connecting the two other nodes from X.

$n = 2$ Here, $|X| = \dim \Pi_2 = 6$. Since $p_{\mathbf{x}}$, for each $\mathbf{x} \in X$, is the product of two lines containing five nodes from X, one of these lines must contain 3 nodes.

The assumption of the GM-conjecture leads to sets X of the following type:
The nodes lie on three lines, which are not concurrent, on each line three nodes, and the intersection points of any two lines belong to X.

The Gasca-Maeztu Conjecture, Proofs

$n=3$ Here, $|X| = \dim \Pi_3 = 10$.

Assuming that $\mathbf{GM_3}$ does not hold, we find that for each node $\mathbf{x} \in X$, the fundamental polynomial $p_{\mathbf{x}}$ factors into three lines each containing three nodes.

Also the fundamental polynomials of two different nodes cannot share a common line.

So, for the ten nodes in X, we need 30 different lines, each one containing three nodes of X.

This is impossible, since there are at most $\binom{10}{2}/\binom{3}{2} = 15$ such lines.

$n=4$ The first proof was given by **Busch** [4].

See also **Carnicer and Gasca** [3], or **HJZ** [5].

$n=5$ This case is treated here; see **HJZ** [1].

The Geometric Condition GC_n

Chung and Yao [6] have introduced this notion:

An n-poised set $X \subset \mathbb{R}^2$ satisfies the **geometric condition $\mathbf{GC_n}$**, if for each $\mathbf{x} \in X$ the set $X \setminus \{\mathbf{x}\}$ is contained in the union of n lines $\ell_1^{\mathbf{x}}, \ldots, \ell_n^{\mathbf{x}}$, say,

$$X \setminus \{\mathbf{x}\} \quad \subset \quad \Gamma_{\mathbf{x}} := \{\ell_1^{\mathbf{x}}, \ldots, \ell_n^{\mathbf{x}}\} \; .$$

This is equivalent to the requirement that X is n-poised, and for each $\mathbf{x} \in X$, the fundamental polynomial is the product of the n linear factors in $\Gamma_{\mathbf{x}}$:

$$p_{\mathbf{x}} = \gamma_{\mathbf{x}} \prod_{j=1}^{n} \ell_j^{\mathbf{x}} \; .$$

Here, $\gamma_{\mathbf{x}} \neq 0$ is a normalization constant enforcing $p_{\mathbf{x}}(\mathbf{x}) = 1$.

Since X is assumed to be poised, the set $\Gamma_{\mathbf{x}}$ is uniquely determined by $\mathbf{x}$, up to normalization of the lines equations.
(Here, and in what follows, we identify a line ℓ with the linear (normalized) polynomial defining the line as its zero set.)

The Natural Lattices by Chung and Yao

These so-called *natural lattices* are the most efficient point lattices satisfying GC_n.

Take $n+2$ lines of $\mathbb{R}^2$ in general position, i.e.,

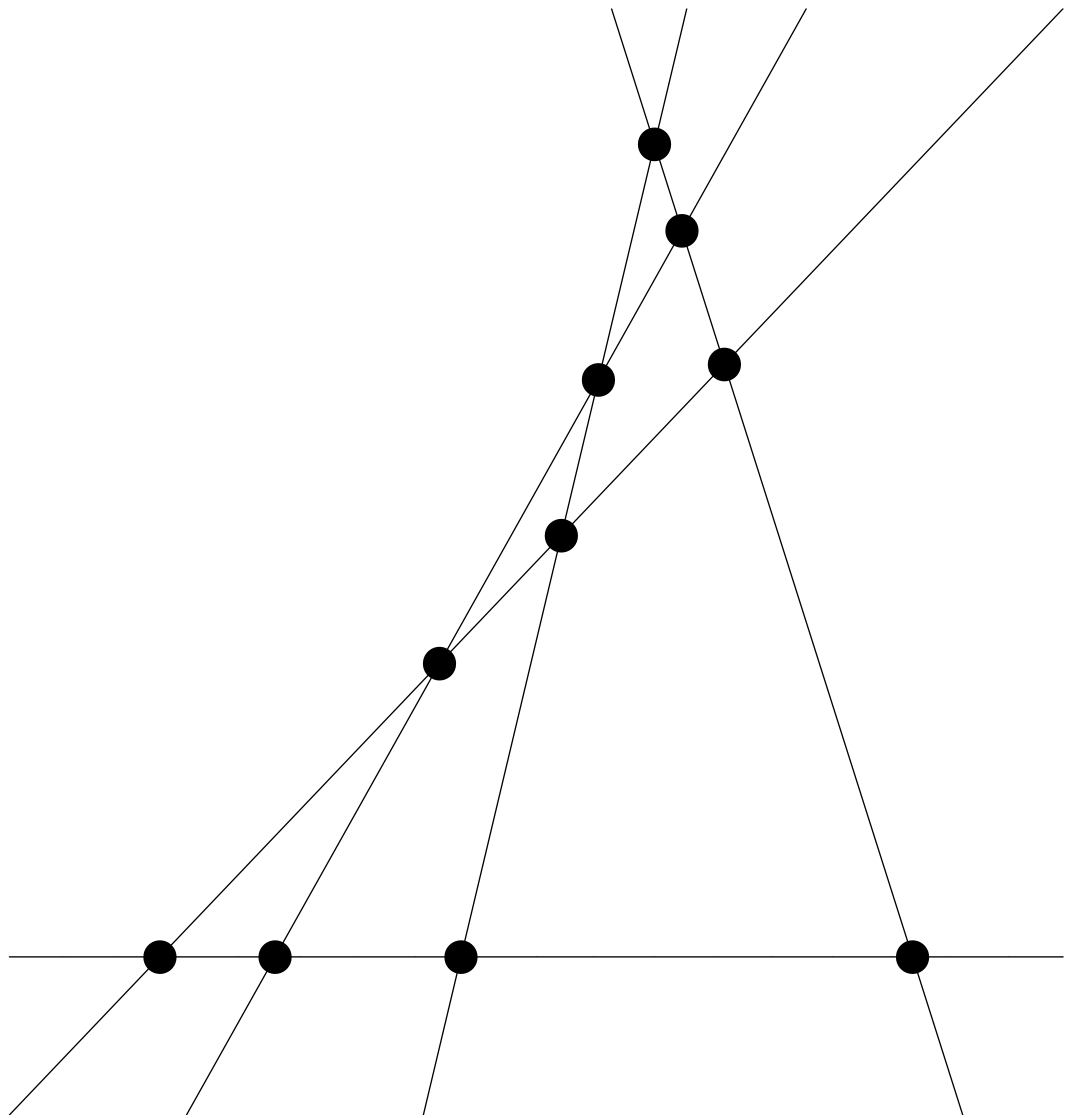
- each pair of lines intersects at a single point, and
- no triple of lines is concurrent.

Then

- the set X of the $\binom{n+2}{2}$ intersection points of all pairs of lines is n-poised, and
- for each $\mathbf{x} \in X$, the set $\Gamma_{\mathbf{x}}$ consists of those n lines not containing $\mathbf{x}$.

For this Chung-Yao natural lattice, each of the $n+2$ lines contains exactly $n+1$ nodes from X.

Example of a Chung and Yao Natural Lattice ($n = 3$)



The Berzolari-Radon Type Sets X

Consider $n+1$ different lines $\ell_1, \ldots, \ell_{n+1}$, and a set X of nodes with $|X| = \binom{n+2}{2}$ such that, for each $k = 1, \ldots, n+1$,

$$n+2-k \text{ nodes of } X \text{ are on } \ell_k \setminus \left(\cup_{j=1}^{k-1} \ell_j \right) \; .$$

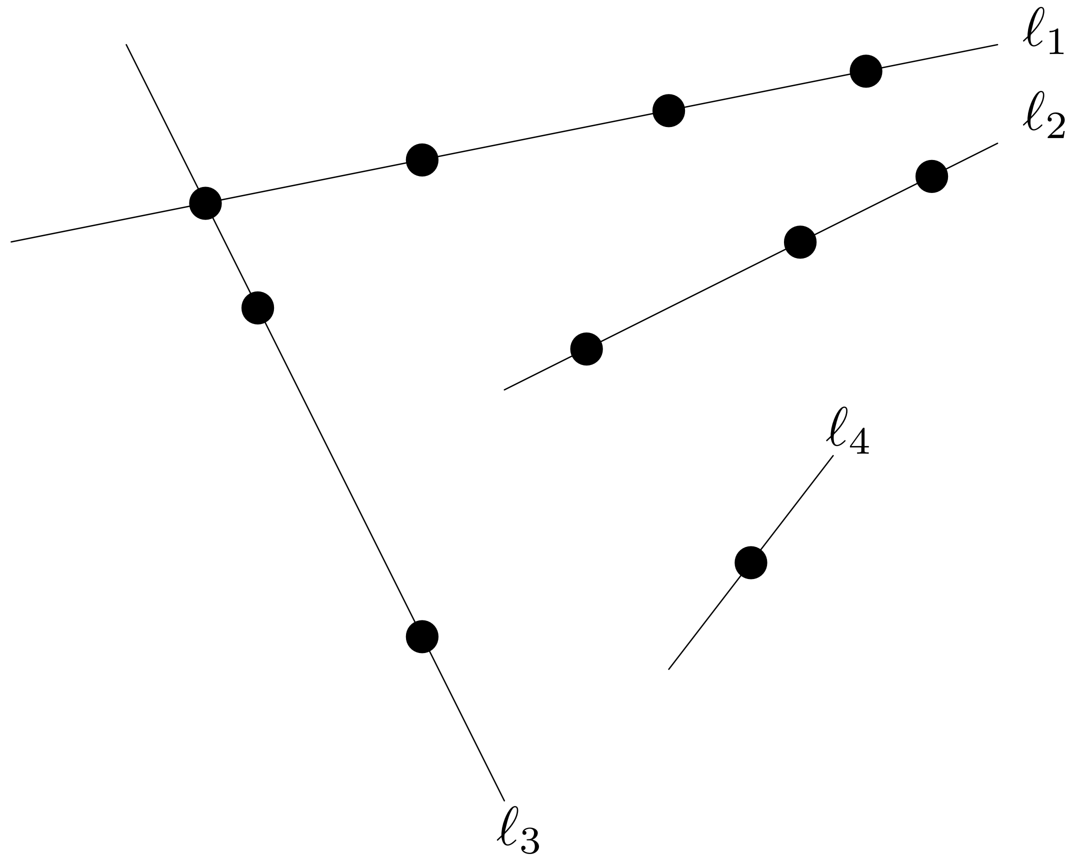
Then X is n-poised, cf. [9], [10].

Such a so-called *Berzolari-Radon set* X does not necessarily satisfy GC_n.

However, at least one point, namely the point $\mathbf{x}$ on ℓ_{n+1} not contained in the former n lines, has a fundamental polynomial which factors according to

$$p_{\mathbf{x}} = \gamma_{\mathbf{x}} \prod_{j=1}^{n} \ell_j \; .$$

Example of a Berzolari-Radon Type Set X ($n = 3$)



Point Distribution of a Berzolari-Radon Type Set

In the former example for $n = 3$, the fundamental polynomial of the last point $\mathbf{x} \in \ell_4$ is the product of the lines ℓ_1, ℓ_2, ℓ_3 which contain the remaining points $X \setminus \{\mathbf{x}\}$ according to the distribution sequence $(4, 3, 2)$.

For general n, the last point $\mathbf{x} \in \ell_{n+1}$ in the construction of a Berzolari-Radon set uses the line sequence

$$(\ell_1, \ell_2, \ell_3, \ldots, \ell_n)$$

according to the point distribution sequence

$$(n + 1, n, n - 1, \ldots, 2) \ .$$

For the Chung-Yao natural lattices, for any $\mathbf{x} \in X$, the used line sequence is the sequence of those n lines not containing $\mathbf{x}$ (in any order), and the corresponding point distribution sequence is again given by

$$(n + 1, n, n - 1, \ldots, 2) \ .$$

Line Sequences and Distribution Sequences for $\mathbf{x} \in X$

Assumption. In what follows, we assume that X is an n-poised GC_n -set.

We use the following notations, for given $\mathbf{x} \in X$:

- $\Gamma_{\mathbf{x}} = \{\ell_1^{\mathbf{x}}, \ldots, \ell_n^{\mathbf{x}}\}$, the (unordered) set of lines (linear polynomials) used in the factorization of the fundamental polynomial $p_{\mathbf{x}}$.

  This set is uniquely determined, up to normalization of the polynomials.

  Notation. We say that $\mathbf{x}$ *uses the lines from* $\Gamma_{\mathbf{x}}$.
- $(\ell_1, \ldots, \ell_n)$ any <u>ordered</u> n-tuple of all lines $\ell_j^{\mathbf{x}} \in \Gamma_{\mathbf{x}}$, is called a *line sequence for* $\mathbf{x}$.
- $(k_1, \ldots, k_n)$, the corresponding *distribution sequence for* $\mathbf{x} \in X$*, is determined by the following count of the nodes of* $X \setminus \{\mathbf{x}\}$:

$$k_i = \left|\ell_i \cap \Big((X \setminus \{\mathbf{x}\}) \setminus \cup_{j<i} \ell_j \Big)\right| , \quad i = 1, \ldots, n \; .$$

Count of Intersection Points in X

So, given a line sequence $(\ell_1, \ldots, \ell_n)$ for $\mathbf{x} \in X$, k_i is the number of nodes from $X \setminus \{\mathbf{x}\}$ lying on ℓ_i, but not on the former lines $\ell_1, \ldots, \ell_{i-1}$.

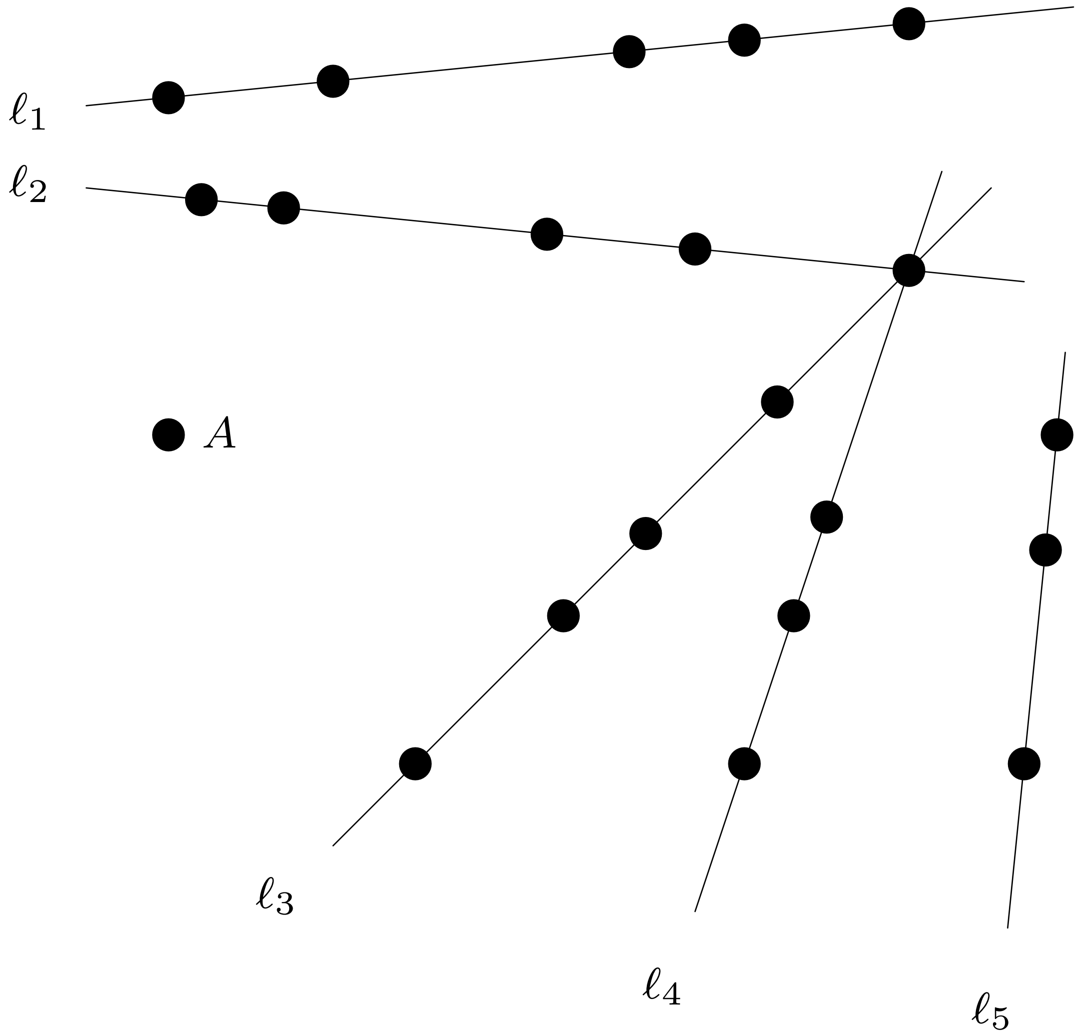
Intersection points of two lines need special consideration:

Notation. If $\{\mathbf{x}\} := \ell_i \cap \ell_j \in X$, for some $i < j$, and if $\mathbf{x}$ is counted by k_i, then $\mathbf{x}$ is called a *primary node for* ℓ_i and a *secondary node for* ℓ_j.

Fact. *The distribution sequence of a node $\mathbf{x} \in X$ is uniquely determined when the line sequence is fixed, but not conversely, and we have*

$$k_1 + \cdots + k_n = \left| X \setminus \{\mathbf{x}\} \right| .$$

A Point, and the Lines Used by It ($n = 5$)



Maximal Distribution Sequence for $\mathbf{x} \in X$

For given $\mathbf{x} \in X$, there are many line sequences and corresponding distribution sequences.

Among the distribution sequences $(k_1, \ldots, k_n)$ for $\mathbf{x}$, there is a unique maximal one (with respect to lexicographic ordering).

We call this the *maximal distribution sequence for* $\mathbf{x}$. It is ordered decreasingly according to

$$n + 1 \geq k_1 \geq k_2 \geq \cdots \geq k_n \geq 2 \ .$$

There may be several line sequences leading to the same maximal distribution sequence of $\mathbf{x}$.

In the former example, the maximal distribution sequence is given by $(5, 5, 4, 3, 3)$, with several corresponding line sequences, e.g.,

$$(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) \ , \quad \text{or } (\ell_3, \ell_1, \ell_2, \ell_5, \ell_4) \ , \quad \text{or } \ \cdots$$

Maximal Lines

Fact 1. *For an n-poised set X, at most $n+1$ points from X can be collinear.*

Otherwise, if a set of nodes $Y \subset X$ with $|Y| > n+1$ were collinear (lying on a line ℓ), we could find a polynomial $p \in \Pi_{n-1}$ vanishing on $X \setminus Y$, since

$$|X \setminus Y| < \binom{n+2}{2} - (n+1) = \binom{n+1}{2} = \text{dim } \Pi_{n-1} \ .$$

The polynomial $\ell \cdot p \in \Pi_n$ would then vanish on all of X contradicting the n-poisedness of the set X.

Notation. (**de Boor** [8]) If $|X \cap \ell| = n+1$, then the line ℓ is said to be *maximal*.

Fact 2. *For an n-poised set X, the existence of a maximal line ℓ implies that ℓ is a divisor of all fundamental polynomials $p_{\mathbf{x}}$, for $\mathbf{x} \in X \setminus \ell$. This means that all those $\mathbf{x}$ use ℓ.*

The $\mathbf{GM_n}$**-conjecture** can now be put as follows:
For an n-poised set satisfying GC_n, there exists a maximal line ℓ and, hence, $X \setminus \ell$ is a GC_{n-1}-set.

Maximal Curves

The idea of a maximal line extends to maximal curves q of degree at most n. Put

$$d(n,k) := \text{dim } \Pi_n - \text{dim } \Pi_{n-k} = \frac{1}{2}\, k\ (\ 2n + 3 - k\ ) \quad \text{for} \quad k \leq n\ .$$

E.g., $d(n,1) = n+1$, $d(n,2) = 2n+1$ and $d(n,3) = 3n$.

Fact. *For an n-poised set X, at most $d(n,k)$ points from X can lie on an algebraic curve q of degree k.*

If such a curve q of degree k contains exactly $d(n,k)$ points from X, then it is again said to be *maximal*.

Such maximal curves q, again, occur as divisors of the fundamental polynomials $p_{\mathbf{x}}$, for $\mathbf{x} \in X \setminus q$.

But in connection with the **GM-Conjecture** we only deal with maximal lines (or reducible maximal curves).

n-dependent Sets

For a (finite) set Y of points, the Vandermonde matrix $\mathcal{V}_n(Y)$ is defined as above as the ($|Y| \times N$)-matrix, $N = \dim \Pi_n$, consisting of the rows

$$[\ \cdots\ \mathbf{y}^\alpha\ \cdots\ ]_{0 \le |\alpha| \le n}\ , \quad \text{for} \quad \mathbf{y} \in Y\ .$$

Here, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ and $|\alpha| := \alpha_1 + \alpha_2$.

Notation. The set Y is said to be n-*dependent*, if the rows of the Vandermonde matrix $\mathcal{V}_n(Y)$ are linearly dependent. Otherwise, Y is said to be n-*independent*.

A useful result is the following:

Theorem 1.9. (**Hakopian and Malinyan** [12]) *A set Y consisting of at most $3n$ nodes is n-dependent if and only if one of the following holds:*

(i) $n + 2 = d(n,1) + 1$ *nodes are collinear.*

(ii) $2n + 2 = d(n,2) + 1$ *nodes belong to a (possibly reducible) conic.*

(iii) $|Y| = 3n = d(n,3)$*, and there exists a* $\gamma \in \Pi_3$ *and* $\sigma \in \Pi_n$ *such that* $Y = \gamma \cap \sigma$.

Cayley-Bacharach

The last statement refers to a special case (m=3) of the following so-called *Cayley-Bacharach theorem*:

Theorem 1.12. *Assume that two algebraic curves of degree m and n, respectively, intersect at $m \cdot n$ points.*

Then the set Y of these intersection points is $(m+n-3)$-dependent.

(Moreover, no point of Y has an $(m+n-3)$-fundamental polynomial.)

A proof of this can be found in **HJZ** [13].

Equivalent Forms of the GM-Conjecture

Theorem. *For given $n \geq 1$, the following are equivalent:*

(i) For $k \leq n$ and any k-poised set X satisfying GC_k, there is a maximal line containing $k+1$ nodes from X.

*(ii) (**Carnicer-Gasca's Three-Line-Theorem**) For $k \leq n$ and any k-poised set X satisfying GC_k, there are three (non concurrent) maximal lines ℓ_1, ℓ_2, ℓ_3, each one containing $k+1$ nodes and altogether containing $3k$ nodes from X. Moreover, for $k \geq 4$, the set $X \setminus \{\ell_1 \cup \ell_2 \cup \ell_3\}$ satisfies GC_{k-3}.*

(iii) For $k \leq n$ and any k-poised set X satisfying GC_k, the maximal sequence of each node $\mathbf{x} \in X$ is given by

$$(k+1, k, k-1, \ldots, 2) \ .$$

$(i) \Rightarrow (ii)$ was proved by Carnicer and Gasca in [7].

$(ii) \Rightarrow (iii)$ is verified in the following way:
At step n, each node in X uses a maximal line ℓ and thus has a maximal distribution sequence of type $(n+1, k_2, \ldots, k_n)$.
The nodes of $X \setminus \ell$ satisfies GC_{n-1}.
Now use induction.

Equivalent Forms of the GM-Conjecture, Continued

$(iii) \Rightarrow (ii)$ Each node $\mathbf{x} \in X$ uses a maximal line $\ell_{\mathbf{x}}$, say.

Fix $\mathbf{x}$, and choose $\mathbf{y} \in \ell_{\mathbf{x}} \cap X$ with maximal line $\ell_{\mathbf{y}}$. Apparently, $\ell_{\mathbf{x}} \neq \ell_{\mathbf{y}}$.
It follows that $\ell_{\mathbf{x}} \cap \ell_{\mathbf{y}} =: \{\mathbf{z}\} \in X$, and $\ell_{\mathbf{z}}$ is the third maximal line, which must be apparently different from both, $\ell_{\mathbf{x}}$ and $\ell_{\mathbf{y}}$.

The property that the first two lines have to intersect at a single node from X, follows from the assumption of X being n-poised.

Otherwise the (reducible) conic $\gamma = \ell_{\mathbf{x}} \cdot \ell_{\mathbf{y}}$ would vanish at $2(n+1)$ points from X, and we could find a polynomial $p \in \Pi_{n-2}$ vanishing at all points from $X \setminus \gamma$, since

$$\left| X \setminus \gamma \right| = \binom{n+2}{2} - 2(n+1) < \binom{n}{2} = \text{dim } \Pi_{n-2} \ .$$

The polynomial $\gamma \cdot p \in \Pi_n$ would vanish at all nodes from X, contradicting the assumption of X being n-poised.

General Idea of Proof of the Conjecture for $n = 5$

The proof uses the structure of possible maximal distribution sequences, and counts the uses of lines.

Assumption A.
– X is 5-poised and satisfies GC_5, and
– no point of $\mathbf{x} \in X$ has a maximal distribution sequence $(k_1, k_2, k_3, k_4, k_5)$ with $k_1 = 6$.

Since $6 > k_1 \geq k_2 \geq k_3 \geq k_4 \geq k_5 \geq 2$ and $k_1 + k_2 + k_3 + k_4 + k_5 = |X| - 1 = 20$, this leaves the following types of maximal distribution sequences:

$$\begin{aligned} &\text{(i)} && (5, 5, 5, 3, 2)\,, \\ &\text{(ii)} && (5, 5, 4, 4, 2)\,, \\ &\text{(iii)} && (5, 5, 4, 3, 3)\,, \\ &\text{(iv)} && (5, 4, 4, 4, 3)\,, \\ &\text{(v)} && (4, 4, 4, 4, 4)\,. \end{aligned}$$

One has to show that none of these cases is possible. (It turns out that it is sufficient to exclude the first two possibilities.)

Count of Used Lines

Actually, the main part of the proof consists in verifying the following table.

total # of nodes on $\widetilde{\ell}$	maximal # of nodes using $\widetilde{\ell}$		
	in general	no node uses $(5,5,5,3,2)$ constellation	no node uses $(5,5,5,3,2)$ or $(5,5,4,4,2)$ constellation
5	6	4	4
4	3	3	2
3	3	1	1
2	1	1	0

The first column distinguishes the possibly used lines $\tilde{\ell}$ according to the number k of nodes lying on them:

$$k := \left| X \cap \tilde{\ell} \right| \in \{5, 4, 3, 2\} \ .$$

(Note that $k = 6$ is excluded by assumption A.)

Lines Used at Least Twice

We quote some intermediate results from the paper:

Proposition 2.11. *Assume that X is a GC_5-set without a maximal line, and suppose that some line $\tilde{\ell}$ is used by two nodes $A, B \in X$.*

Then $\tilde{\ell}$ passes through <u>at least three</u> nodes of X.

If $\tilde{\ell}$ passes through <u>exactly three</u> nodes, then there exists a third node C using $\tilde{\ell}$.
Furthermore, A, B and C share three other lines, each passing through five primary nodes.
For each of the three nodes, the maximal distribution sequence is $(5, 5, 5, 3, 2)$, and the other two points are the primary points on the respective fifth line.
In particular, $\tilde{\ell}$ is used exactly three times.

Lines Used at Least Three Times

Proposition 2.12. *Assume that X is a GC_5-set without a maximal line, and suppose that some line $\tilde{\ell}$ is used by three nodes $A, B, C \in X$.*

Then $\tilde{\ell}$ passes through <u>at least three nodes</u> of X.

If $\tilde{\ell}$ passes through <u>exactly three</u> nodes then Proposition 2.11 applies, so in particular, it is used exactly three times.

If $\tilde{\ell}$ passes through <u>exactly four</u> nodes, then A, B and C share $\tilde{\ell}$ and three other lines, ℓ_2 and ℓ_3 passing through five, and ℓ_4 passing through four primary nodes.
For each of the three nodes its maximal distribution sequence is $(5, 5, 4, 4, 2)$ or $(5, 5, 5, 3, 2)$ where the latter case occurs if $\tilde{\ell}$ and ℓ_4 intersect at a node of X.
In each case, the respective other two nodes are the primary nodes on the respective fifth line, so $\tilde{\ell}$ can only be used by A, B and C, i.e., it is used exactly three times.

Lines Used at Least Four Times

The following result immediately follows from the previous one:

Corollary 2.13. *If some line $\tilde{\ell}$ is used by four nodes in X, then it has to pass through five nodes of X.*

Lines Used at Least Five Times

Proposition 2.12. *Assume that* X *is a* GC_5*-set without a maximal line, and suppose that some line* $\tilde{\ell}$ *is used by five nodes in* X*.*

Then $\tilde{\ell}$ *passes through five nodes of* X*, and it is actually used by* <u>*exactly six*</u> *nodes in* X*. These six nodes form a* GC_2*-set and share two more lines with five primary nodes each, i.e., each of these six nodes has the maximal distribution sequence* $(5, 5, 5, 3, 2)$*.*

Case $(5, 5, 5, 3, 2)$

The hardest part in the proof of the conjecture for $n = 5$ is the verification of the following

Proposition 3.12. *Assume that X is a GC_5-set without a maximal line. Then for no point of X, the maximal distribution sequence is $(5, 5, 5, 3, 2)$.*

Our proof needs 8 pages, plus some additional references to other supplementary results, including our $n = 4$ paper in JAT.

A corollary of this result is the following:

Lemma 3.13. *Assume that X is a GC_5-set without a maximal line. Then the following hold:*

- *(i) No line contains exactly three nodes of X, and for $m = 2, 4$ and 5, any line containing exactly m nodes of X is used exactly $m - 1$ times.*
- *(ii) No two lines used by the same node intersect at a point in X (i.e., there are no secondary nodes).*

Case $(5, 5, 4, 4, 2)$

Another 4 pages argument shows

Proposition 3.17. *Assume that X is a GC_5-set without a maximal line. Then for no point of X, the maximal distribution sequence is $(5, 5, 4, 4, 2)$.*

The proof of this result, and of Proposition 3.12., uses a result from HJZ[5] counting the maximal possible number M_{k_1,k_2,k_3} of lines passing through one point each of three disjoint subsets X_1, X_2 and X_3 of X satisfying $|X_i| = k_i$, $i = 1, 2, 3$, subject to the following additional assumptions:

- X_1 and X_2 are collinear (not necessarily X_3), $X_1 \subset \ell_1$ and $X_2 \subset \ell_2$, say, with $\ell_1 \neq \ell_2$.
- The intersection point of ℓ_1 and ℓ_2 is not in $X_1 \cup X_2$, and none of the points in X_3 lies on one of these two lines.

In [5], we have shown that $M_{4,4,4} = 12$ and $M_{5,5,5} = 19$.

Finishing the Proof of the Conjecture for $n = 5$

Assume that the last column applies for the count of uses of a line $\tilde{\ell}$.

Fix any point $A \in X$, and consider all lines through A and at least one other point of X. Let $n_m(A)$ be the number of such lines containing exactly m points from X, including A. This gives

$$1 \cdot n_2(A) + 2 \cdot n_3(A) + 3 \cdot n_4(A) + 4 \cdot n_5(A) = \Big| X \setminus \{A\} \Big| = 20 \ .$$

The last column in our table tells that the total uses of these lines through A by points from $X \setminus \{A\}$ can be at most

$$M := 0 \cdot n_2(A) + 1 \cdot n_3(A) + 2 \cdot n_4(A) + 4 \cdot n_5(A) \ ,$$

while $M \geq 20$, since each point from $X \setminus \{A\}$ must use at least one such line.

We arrive at the condition $n_2(A) = n_3(A) = n_4(A) = 0$ and $n_5(A) = 5$.

This tells that the 21 nodes from X are situated on five lines, a contradiction to the 5-poisedness of the set.

References

The talk is based on

[1] H. Hakopian, K. Jetter and G. Zimmermann,
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The conjecture was stated in

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See also

[3] J. M. Carnicer and M. Gasca,
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The first proof of the Gasca-Maeztu conjecture for the case $n = 4$ was given by

[4] J. R. Busch,
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See also the proof by Carnicer and Gasca in [3], and

[5] H. Hakopian, K. Jetter and G. Zimmermann,
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The so-called geometric characterization of point constellations - where the GM conjecture refers to - was introduced by

[6] K. C. Chung and T. H. Yao,
On lattices admitting unique Lagrange interpolations, SIAM J. Numer. Anal. 14 (1977), 735–743.

See also

[7] J. M. Carnicer and M. Gasca,
On Chung and Yao's geometric characterization for bivariate polynomial interpolation, in Saint Malo Proceedings 2002, Nashboro Press, Brentwood (2003), pp. 21–30.

where the Three-Line-Theorem was proved.

Concerning generalizations of this, see

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The Berzolari-Radon construction of n-poised sets X appears in

[9] L. Berzolari,
Sulla determinazione d'una curva o d'una superficie algebrica a su alcune questioni di postulazione, Ist. Lomb. Rend. (II. Ser.) 47 (1914), 556–564.

and was used in the construction of cubature formulas as in

[10] J. Radon,
Zur mechanischen Kubatur, Monatshefte Math. 52 (1948), 286–300.

The construction was reinvented several times.

An early instant of constructing such cubature formulas can be found in

[11] O. Biermann,
Über näherungsweise Cubaturen, Monats. Math. Phys. 14 (1903), 211–225.

References

Useful papers on n-dependent sets are

[12] H. Hakopian and A. Malinyan,
Characterization of n-independent sets of $\leq 3n$ points, Jaén J. Approx. 4 (2012), 119–134.

[13] H. Hakopian, K. Jetter and G. Zimmerann,
Vandermonde matrices for intersection points of curves, Jaén J. Approx. 1 (2009), 67–81.

It may be advisable, to have a short look at an old note of mine:

[14] K. Jetter,
Some contributions to bivariate interpolation and cubature, in: Approximation Theory IV, 1983, pp. 533–538.