
Topics in Computer Aided
Geometric Design,
Erice May 12-19, 1990

Smooth Simplex Splines for the Powell-Sabin 12-split

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Outline

I: Simplex splines

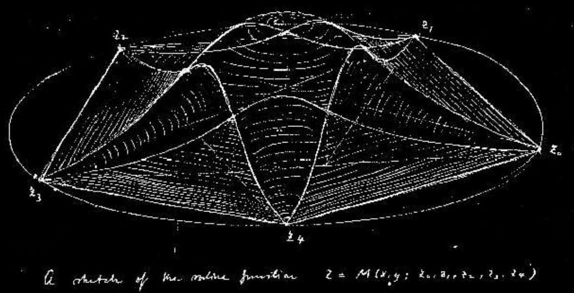
II: A quadratic simplex spline basis for PS12 on one triangle

III: Higher degree splines on the 12-split

PART I:
simplex splines

Schoenberg's View of the Bivariate B-Spline

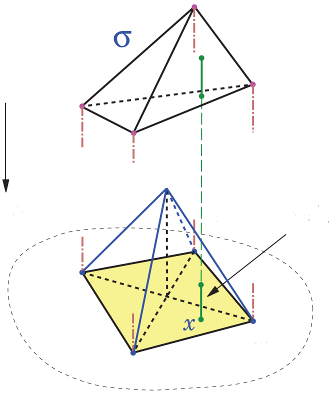
In a letter from Iso Schoenberg to Phillip Davis from 1965:



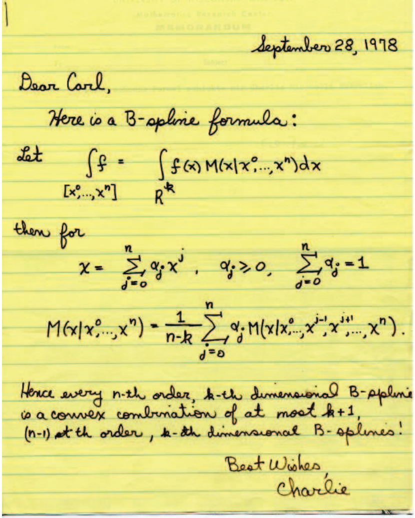
"A sketch of the spline function
$z = M(x, y; z_0, z_1, z_2, z_3, z_4)$"

Simplex spline definitions

- geometric
- variational
- recurrence relation



Recurrence Relation



September 28, 1978

Dear Carl,

Here is a B-spline formula:

Let $\int_{[x^0,\dots,x^n]} f = \int_{R^k} f(x)\, M(x|x^0,\dots,x^n)\,dx$

then for

$x = \sum_{j=0}^{n} \alpha_j x^j, \quad \alpha_j \geq 0, \quad \sum_{j=0}^{n} \alpha_j = 1$

$M(x|x^0,\dots,x^n) = \frac{1}{n-k} \sum_{j=0}^{n} \alpha_j M(x|x^0,\dots,x^{j-1},x^{j+1},\dots,x^n).$

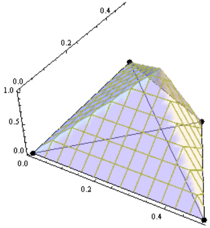
Hence every n-th order, k-th dimensional B-spline is a convex combination of at most k+1, (n-1) st th order, k-th dimensional B-splines!

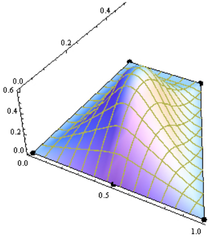
Best Wishes
Charlie

$$\int_{[x0,\dots,x^n]} f := \int_{\Delta^n} f(v_0 x^0 + v_1 x^1 + \cdots + v_n x^n)\,dv_1 \cdots dv_n$$

Bivariate simplex spline properties

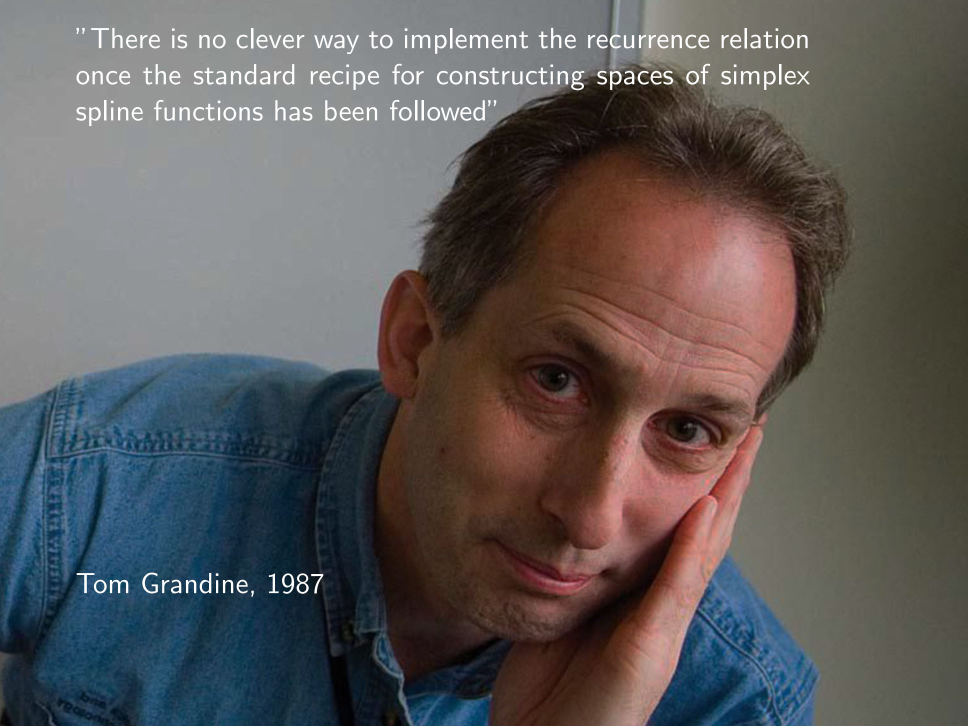
- Let **X** be a collection of $d+3$ points $\mathbf{x}_1, \ldots, \mathbf{x}_{d+3}$ in $\mathbb{R}^2$
- A simplex spline $S = S[\mathbf{X}] : \mathbb{R}^2 \to \mathbb{R}$ with **knots X** is a nonnegative piecewise polynomial
- the **degree** is d
- the **support** is the convex hull of **X**
- the **knotlines** are the edges in the complete graph of **X**
- a knot line has **multiplicity** m if it contains $m+1$ of the points in **X**
- $S \in C^{d-m}$ across a knotline of multiplicity m





Some simplex spline spaces

- **Triangulate a slab**, de Boor, 1976.
- **Complete Configurations**, Hakopian[1981], Dahmen, Michelli[1983],
- **Pull apart**, Dahmen, Micchelli[1982], Höllig[1982], Dahmen, Micchelli,Seidel[Erice 1990],
- **Delaunay configurations**, Neamtu[2000-2007]


"There is no clever way to implement the recurrence relation once the standard recipe for constructing spaces of simplex spline functions has been followed"
Tom Grandine, 1987

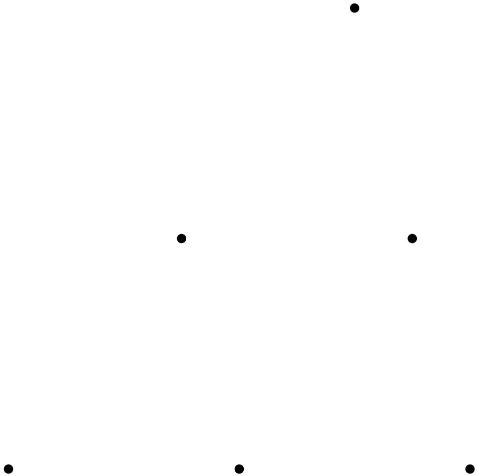
What should be the space of Simplex splines on a triangulation?

PART II:

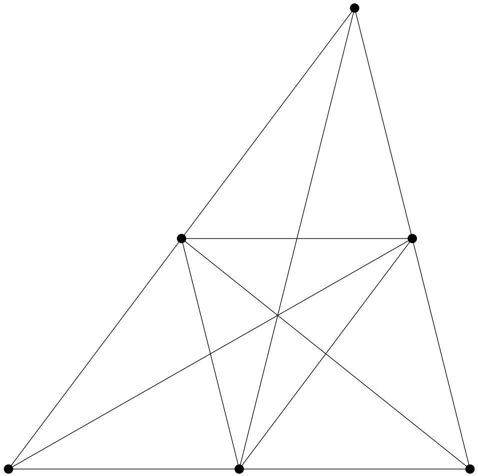
A Simplex spline basis for PS12 on one triangle

Cohen, E., T. Lyche, and R. F. Riesenfeld, A B-spline-like basis for the Powell-Sabin 12-split based on simplex splines, Math. Comp., 82(2013), 1667-1707

The PS12-split (Powell,Sabin 1977)



The PS12-split



The PS12 spline space

$\mathbb{S}_2^1($$) = \{f \in C^1($$) : f_{|_{\Delta_i}} \in \Pi_2,\ i = 1, \ldots, 12\}$

$$\dim(\mathbb{S}_2^1(\Delta_{PS12})) = 12$$

Computing with PS12

- Bernstein-Bézier methods,
- FEM nodal basis, (Oswald)
- minimal determining set (Alfeld, Schumaker, Sorokina)
- subdivision (Dyn, Lyche, Davydov, Yeo)
- quadratic S(implex) - splines (Cohen, Lyche, Riesenfeld)

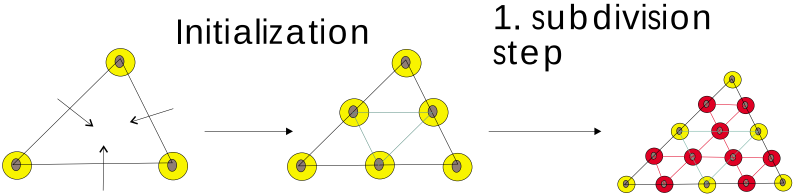
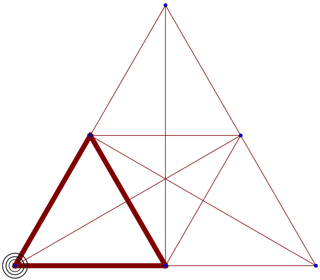
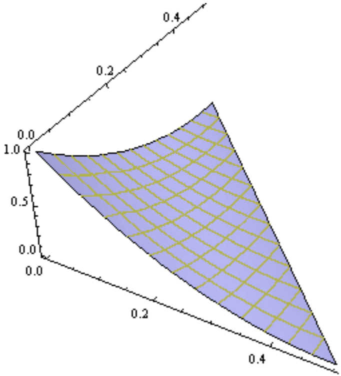


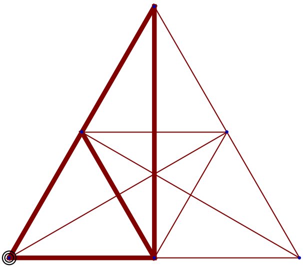
Fig. 2. Subdividing the PS-12 split element. A circle around a vertex means that both the function value and the gradient are known at that vertex.

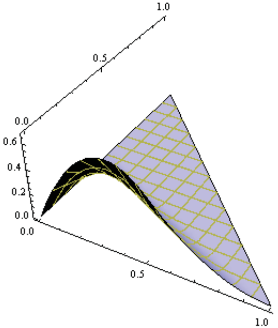
3 corner S-splines for the quadratic case; support 1/4



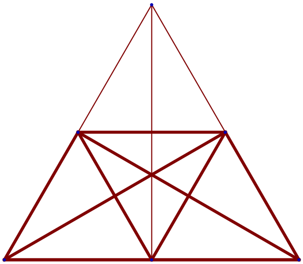


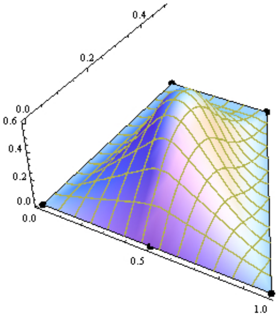
6 half support S-splines for the quadratic case





3 trapezoidal support S-splines for the quadratic case





Properties

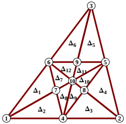
- Local linear independence,
- nonnegative partition of unity,
- stable recurrence relations,
- fast pyramidal evaluation algorithms,
- differentiation formula,
- L_p stable basis,
- subdivision algorithms of Oslo- and Lane,Riesenfeld type,
- quadratic convergence of control mesh,
- well conditioned collocation matrices for Lagrange and Hermite interpolation,
- explicit dual functionals,
- dual polynomials and Marsden-like identity.

Marsden-like identity

Univariate quadratic: $(1-yx)^2 = \sum_j B_{j,2}(x)(1-yt_{j+1})(1-yt_{j+2})$

Quadratic S-splines: $\mathbf{x} \in \Delta$, $\mathbf{y} \in \mathbb{R}^2$

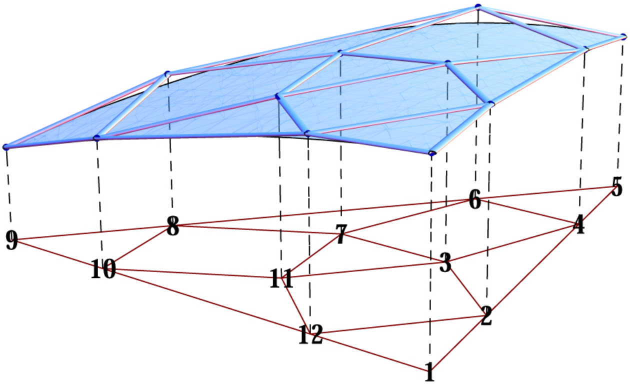
$$(1-\mathbf{y}^T\mathbf{x})^2 = \sum_{j=1}^{12} S_{j,2}(\mathbf{x})(1-\mathbf{y}^T\mathbf{p}_{j,1}^*)(1-\mathbf{y}^T\mathbf{p}_{j,2}^*).$$



$$[\mathbf{p}_{1,1}^*, \ldots, \mathbf{p}_{12,1}^*] := [\mathbf{p}_1, \mathbf{p}_1, \mathbf{p}_4, \mathbf{p}_4, \mathbf{p}_2, \mathbf{p}_2, \mathbf{p}_5, \mathbf{p}_5, \mathbf{p}_3, \mathbf{p}_3, \mathbf{p}_6, \mathbf{p}_6],$$
$$[\mathbf{p}_{1,2}^*, \ldots, \mathbf{p}_{12,2}^*] := [\mathbf{p}_1, \mathbf{p}_4, \mathbf{p}_{10}, \mathbf{p}_2, \mathbf{p}_2, \mathbf{p}_5, \mathbf{p}_{10}, \mathbf{p}_3, \mathbf{p}_3, \mathbf{p}_6, \mathbf{p}_{10}, \mathbf{p}_1],$$

- $1 = \sum_{j=1}^{12} S_{j,2}(\mathbf{x}), \quad \mathbf{x} \in \Delta,$
- $\mathbf{x} = \sum_{j=1}^{12} S_{j,2}(\mathbf{x})\mathbf{m}_j, \quad \mathbf{x} \in \Delta, \quad \mathbf{m}_j := (\mathbf{p}_{j,1}^* + \mathbf{p}_{j,2}^*)/2.$

domain- and control mesh



The control points are at a distance $O(h^2)$ from the surface, where h is the longest side of the triangle..

Dual functionals

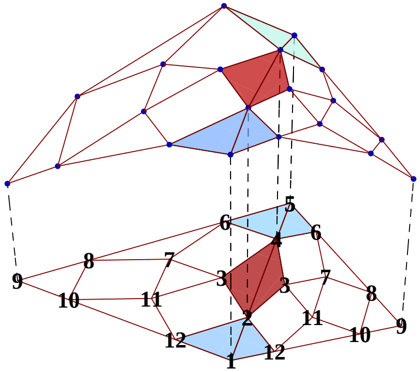
Univariate quadratic:

$$\lambda_j f := 2f(t_{j+3/2}) - \frac{1}{2}f(t_{j+1}) - \frac{1}{2}f(t_{j+2}), \quad \lambda_i B_{j,2} = \delta_{ij}$$

Quadratic S-splines:

$$\lambda_j f := 2f(\mathbf{m}_j) - \frac{1}{2}f(\mathbf{p}^*_{j,1}) - \frac{1}{2}f(\mathbf{p}^*_{j,2}) \quad \lambda_i S_{j,2} = \delta_{ij}$$

C^1 smoothness is controlled as in the polynomial Bézier case.

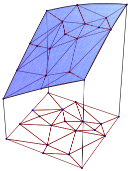


PART III:

Higher degree splines on the 12-split

Joint work with Georg Muntingh

Smooth splines on triangulation



- Consider a general triangulation in the plane
- consider a subdivided triangulation with the 12-split on each triangle and a piecewise polynomial of degree d on this triangulation
- $d = 2$ necessary and sufficient for C^1
- $d = 5$ necessary and sufficient for C^2

Dimensions of $\mathbb{S}_d^r(\triangle)$

$$\mathbb{S}_d^r(\triangle) = \{f \in C^r(\triangle) : f_{|_{\Delta_i}} \in \Pi_d,\ i = 1, \ldots, 12\}$$

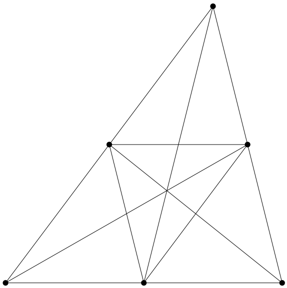
Theorem

For any integers d, r with $d \geq 0$ and $d \geq r \geq -1$

$$\dim \mathbb{S}_d^r(\triangle) = \frac{1}{2}(r+1)(r+2) + \frac{9}{2}(d-r)(d-r+1) + \frac{3}{2}(d-2r-1)(d-2r)_+ + \sum_{j=1}^{d-r}(r-2j+1)_+, \tag{1}$$

where $z_+ := \max\{0, z\}$ for any real z.

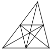
Proof



One cell and three flaps. Use cell dimension formula in Lai-Schumaker book.

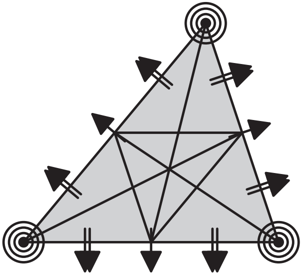
Dimensions of $\mathbb{S}_d^r($$)$

d/r	−1	0	1	2	3	4	5	6	7	8	9	10
0	12	1										
1	36	10	3									
2	72	31	12	6								
3	120	64	30	16	10							
4	180	109	60	34	21	15						
5	252	166	102	61	39	27	21					
6	336	235	156	100	66	46	34	28				
7	432	316	222	151	102	73	54	42	36			
8	540	409	300	214	150	109	81	63	51	45		
9	660	514	390	289	210	154	117	91	73	61	55	
10	792	631	492	376	282	211	162	127	102	84	72	66
11	936	760	606	475	366	280	216	172	138	114	96	84

An interesting family on 

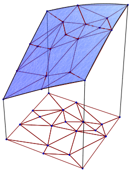
- For any positive integer n consider on  the spline space $\mathbb{S}^{2n-1}_{3n-1}($$)$ of splines of smoothness $2n-1$ and degree $3n-1$.
- $n = 1$: C^1 quadratics
- $n = 2$: C^3 quintics
- $n = 3$: C^5 octic
- $\dim \mathbb{S}^{2n-1}_{3n-1}($$) = \frac{15}{2}n^2 + \frac{9}{2}n.$

Hermite degrees of freedom $\mathbb{S}^3_5(\triangle)$



- $\dim \mathbb{S}^3_5(\triangle) = 39$
- 10 derivatives at 3 corners
- 3 first order cross boundary derivatives
- 6 second order cross boundary derivatives
- Connects to neighboring triangles with smoothness C^2.

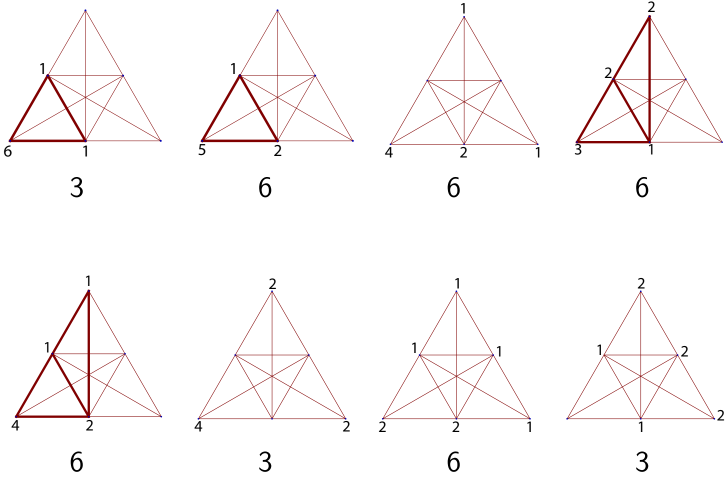
Spline space on triangulation of smoothness C^n



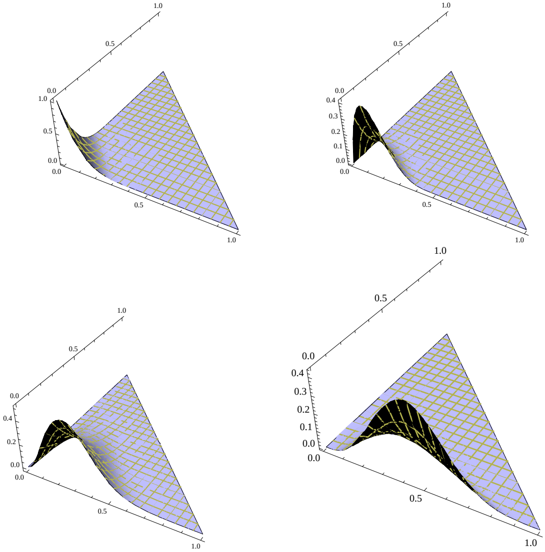
- Consider a triangulation in the plane
- use $\mathbb{S}^{2n-1}_{3n-1}(\triangle)$ on each triangle
- get a global spline space of smoothness C^n
- for $n = 2$ we get a C^2 spline space of dimension $10V + 3E$
- for $n \geq 1$ we get a C^n spline space of dimension

$$n(2n+1)V + \frac{1}{2}n(n+1)E$$

Simplex spline basis for $\mathbb{S}_5^3(\triangle)$



Simplex spline basis for $\mathbb{S}_5^3(\triangle)$



Simplex spline basis for $\mathbb{S}^3_5(\triangle)$;

- nonnegative partition of unity
- globally linearly independent
- can be computed recursively
- reduces to univariate quintic B-splines on boundary
- not locally linearly independent
- no simplex spline basis for for $\mathbb{S}^3_5(\triangle)$ that is locally linearly independent

Thank you!

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- Francis Bach (INRIA, France)
- Robert Ghrist (Pennsylvania Univ., USA)
- Lars Grasedyck (RWTH Aachen, Germany)
- Leonidas Guibas (Stanford Univ., USA)
- Kai Hormann (Lugano Univ., Switzerland)
- Pencho Petrushev (Univ. South Carolina, USA)
- Gabriel Peyre (Ceremade, Paris-Dauphine)
- Konrad Polthier (Univ. Berlin, Germany)
- Rebecca Willett (Duke Univ., USA)
- Grady Wright (Boise State Univ., USA)
