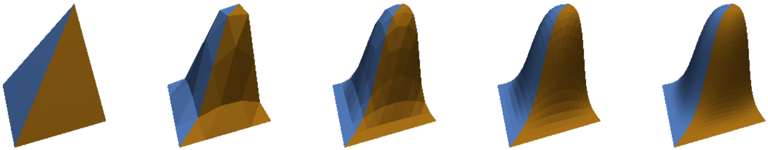


A Hermite Subdivision Scheme for Smooth Macro-Elements on the Powell-Sabin-12 Split

Georg Muntingh, SINTEF, Oslo

Joint work with Tom Lyche and Nelly Villamizar

OUTLINE

I: Macro-Elements on the 12-Split

II: A Hermite Subdivision Scheme

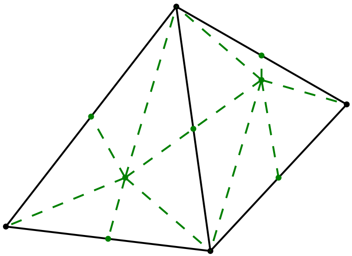
III: Numerical Experiments

PART I:

Macro-Elements on the 12-Split

THE POWELL-SABIN SPLITS

6-split



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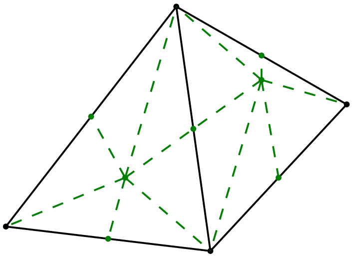
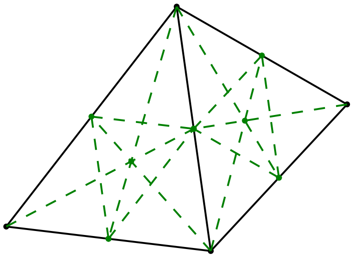
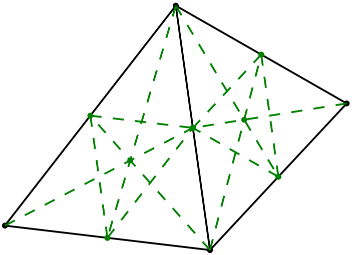
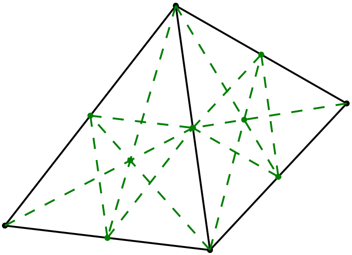


12-split





SPLINES ON THE 12-SPLIT



The space $\mathcal{S}_d^r(\Delta_{12})$ of splines of degree d and smoothness C^r on the 12-split has been studied by several authors:

- Powell and Sabin (1977)
- Schumaker and Sorokina (2005)
- Alfeld and Schumaker
- Speleers
- etc.

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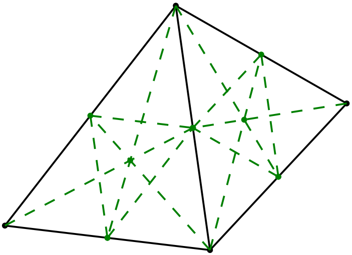

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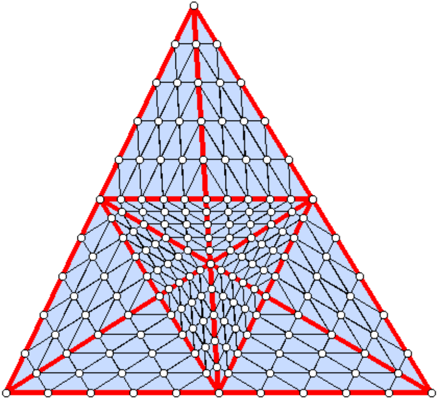
Questions:

- What is a "good" basis for $\mathcal{S}_d^r$?
- Can they be evaluated quickly?

spline spaces on the 12-split

SPLINE SPACES ON THE 12-SPLIT

l.b. = 39 = 39 = 39 = u.b.



r = 3 d = 5 dim = 39 MDS so far: 0

DIMENSION FORMULA FOR THE 12-SPLIT

$\dim \mathcal{S}_d^r(\Delta_{12})$	C^{-1}	C^0	C^1	C^2	C^3	C^4	C^5
$d = 0$	12	1					
$d = 1$	36	10	3				
$d = 2$	72	31	**12**	6			
$d = 3$	120	64	30	16	10		
$d = 4$	180	109	60	34	21	15	
$d = 5$	252	166	102	61	**39**	27	21

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THEOREM

Let $z_+ := \max\{0, z\}$. *For any* $d, r \in \mathbb{Z}$ *with* $d \geq 0$ *and* $d \geq r \geq -1$,

$$\dim \mathcal{S}_d^r(\Delta_{12}) = \frac{1}{2}(r+1)(r+2) + \frac{9}{2}(d-r)(d-r+1) + \frac{3}{2}(d-2r-1)(d-2r)_+ + \sum_{j=1}^{d-r}(r-2j+1)_+$$

MACRO-ELEMENTS ON THE 12-SPLIT

macro-elements on the 12-split

Definition (macro-element)

A **macro-element** defined on a triangle T consists of a finite-dimensional linear space $\mathcal{S}$ of functions defined on T, and a set Λ of linear functionals forming a basis for the dual of $\mathcal{S}$.

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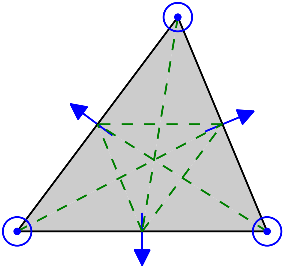

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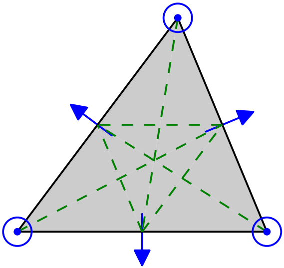
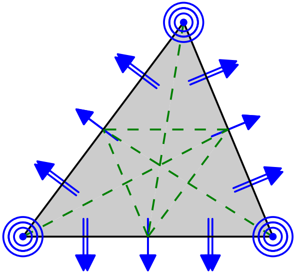


C^3 quintics



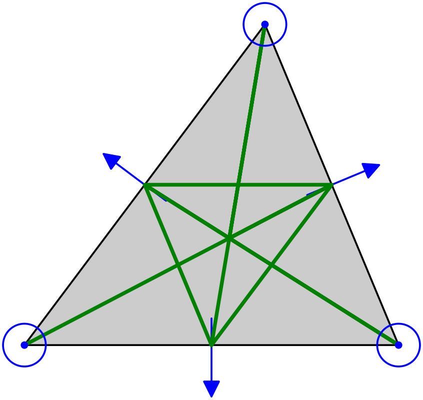
etc.

PART II:

A Hermite Subdivision Scheme

A SUBDIVISION SCHEME FOR C^1 QUADRATICS

Dyn and Lyche (1998) described a Hermite subdivision scheme for evaluating C^1 quadratics on the 12-split.



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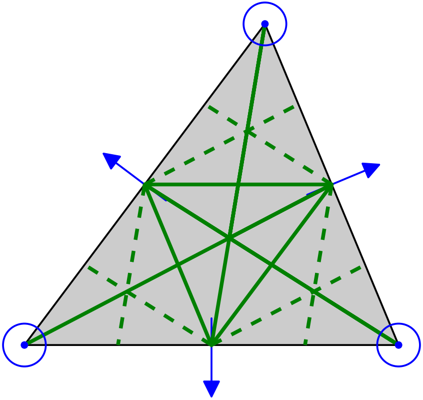
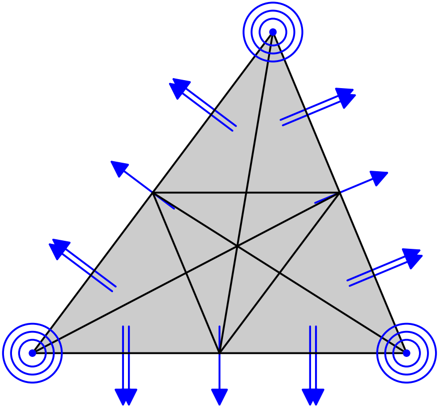
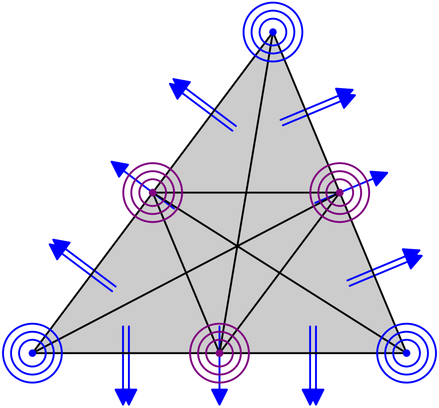


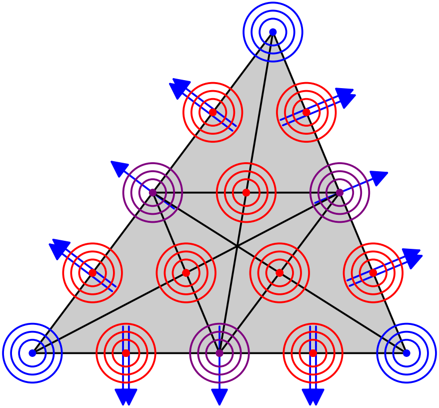
WHAT ABOUT THE C^3 QUINTICS?



C^3 QUINTICS: **INITIALIZATION**



C^3 QUINTICS: INITIALIZATION + SUBDIVISION



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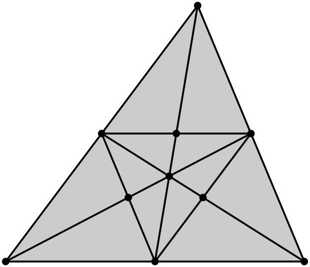

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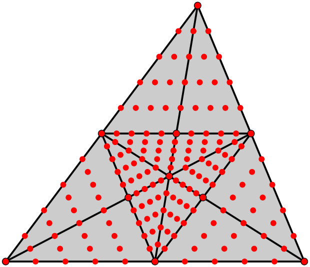
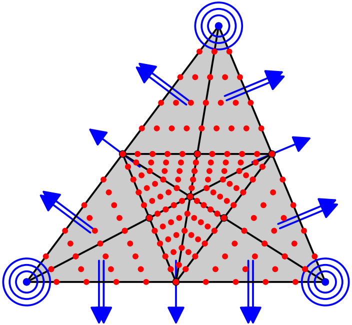
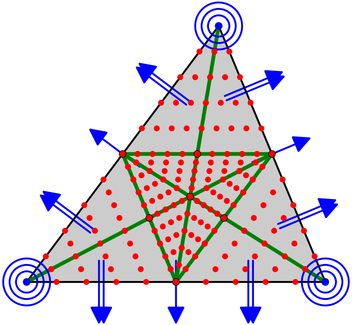


$12 \times 21 = 252$ unknown B-coefficients.



$12 \times 21 = 252$ unknown B-coefficients.

$3 \times (10 + 3) = 39$ initial conditions.



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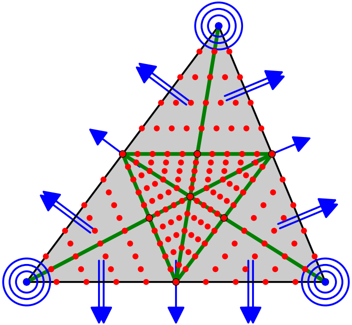

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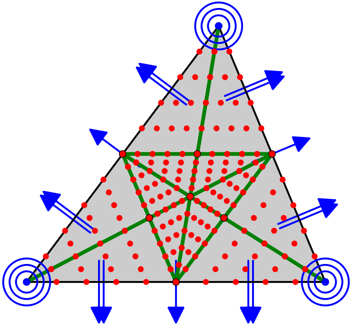

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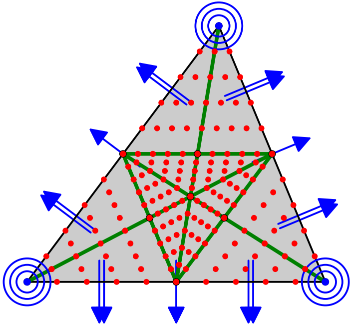
$$D_{\mathbf{u}_m} \cdots D_{\mathbf{u}_1} p(\mathbf{v}) = \frac{d!}{(d-m)!} \sum_{i+j+k=d-m} c_{ijk}^{(m)} B_{ijk}^{d-m}(\mathbf{v}),$$



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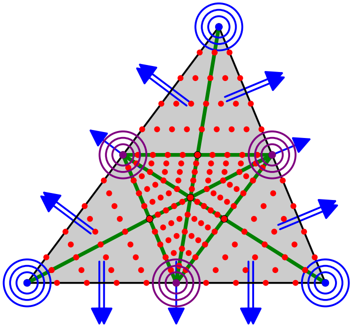

INITIALIZATION: EXPLICIT FORMULAS

With $t = t_C$ and $m = m_C$:

$$
\begin{aligned}
f_{AB} =& \frac{1}{2} f_{A+B} + \frac{7}{40} f^{t}_{A-B} + \frac{1}{40} f^{tt}_{A+B} + \frac{1}{640} f^{ttt}_{A-B} \\
f^{t}_{AB} =& -\frac{5}{2} f_{A-B} - \frac{3}{4} f^{t}_{A+B} - \frac{3}{32} f^{tt}_{A-B} - \frac{1}{192} f^{ttt}_{A+B} \\
f^{m}_{AB} =& \text{ given} \\
f^{tt}_{AB} =& -2 f^{t}_{A-B} - \frac{1}{2} f^{tt}_{A+B} - \frac{1}{24} f^{ttt}_{A-B} \\
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f^{tmm}_{AB} =& -8 f^{mm}_{AAB-ABB} + 4 f^{mm}_{A-B} + \frac{1}{2} f^{tmm}_{A+B}
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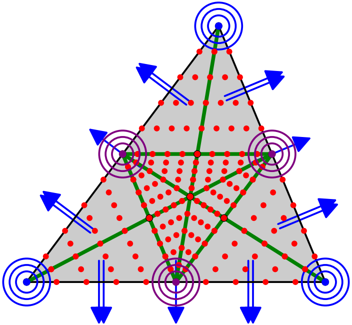
$$\begin{aligned} f^{mmm}_{AB} = \; & + 45 f_{A+B} + 36 f^{t}_{A-B} - 108 f^{m}_{AB} \\ & + 15 f^{mm}_{AAB+ABB} + 45 f^{m}_{A+B} - \frac{217}{16} f^{mm}_{A+B} \\ & + \frac{25}{64} f^{mmm}_{A+B} + \frac{153}{16} f^{tm}_{A-B} - \frac{251}{128} f^{tmm}_{A-B} \\ & + \frac{567}{64} f^{tt}_{A+B} + \frac{43}{256} f^{ttm}_{A+B} + \frac{303}{512} f^{ttt}_{A-B} \\ & + 48 (f^{m_A}_{BC} + f^{m_B}_{CA}) + f^{m_A m_A}_{BCC} \\ & + f^{m_B m_B}_{CCA} - 7 (f^{m_A m_A}_{BBC} + f^{m_B m_B}_{CAA}) \\ & - 90 f_C - 24 f^{m}_C - \frac{23}{8} f^{mm}_C \\ & - \frac{5}{32} f^{mmm}_C - \frac{135}{32} f^{tt}_C - \frac{79}{128} f^{ttm}_C \end{aligned}$$

SUBDIVISION



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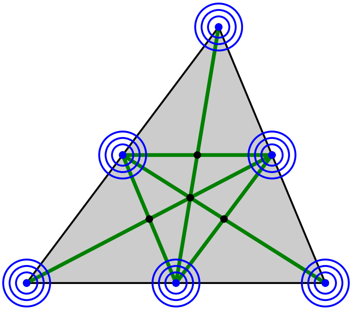

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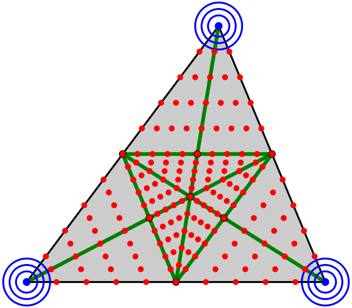

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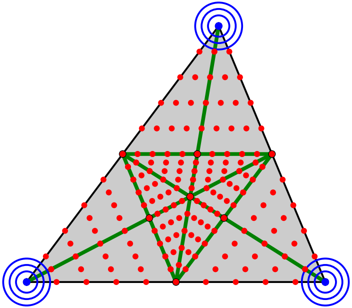

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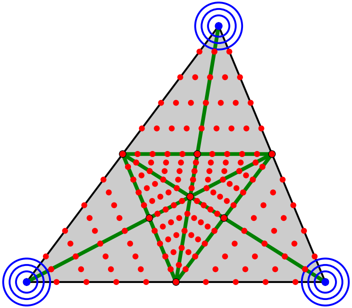

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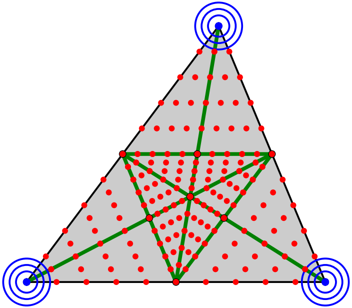

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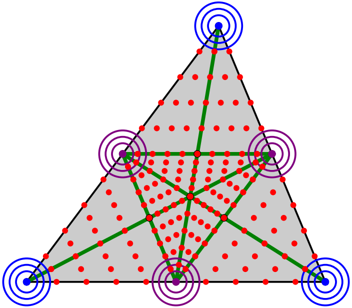

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$$f^{ttm}_{AB} = -48 f^{m}_{AB} + 24 f^{m}_{A+B} + 6 f^{tm}_{A-B} + \frac{1}{2} f^{ttm}_{A+B}$$

$$f^{tmm}_{AB} = -8 f^{mm}_{AAB-ABB} + 4 f^{mm}_{A-B} + \frac{1}{2} f^{tmm}_{A+B}$$

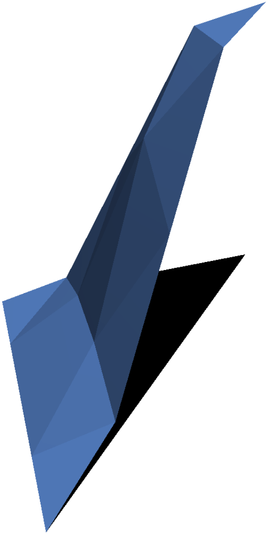
$$\begin{aligned} f^{mmm}_{AB} = & \\ & + 45 f_{A+B} + 36 f^{t}_{A-B} - 108 f^{m}_{AB} \\ & + 15 f^{mm}_{AAB+ABB} + 45 f^{m}_{A+B} - \frac{217}{16} f^{mm}_{A+B} \\ & + \frac{25}{64} f^{mmm}_{A+B} + \frac{153}{16} f^{tm}_{A-B} - \frac{251}{128} f^{tmm}_{A-B} \\ & + \frac{567}{64} f^{tt}_{A+B} + \frac{43}{256} f^{ttm}_{A+B} + \frac{303}{512} f^{ttt}_{A-B} \\ & + 48 (f^{m_A}_{BC} + f^{m_B}_{CA}) + f^{m_A m_A}_{BCC} \\ & + f^{m_B m_B}_{CCA} - 7 (f^{m_A m_A}_{BBC} + f^{m_B m_B}_{CAA}) \\ & - 90 f_C - 24 f^{m}_C - \frac{23}{8} f^{mm}_C \\ & - \frac{5}{32} f^{mmm}_C - \frac{135}{32} f^{tt}_C - \frac{79}{128} f^{ttm}_C \end{aligned}$$

PART III:

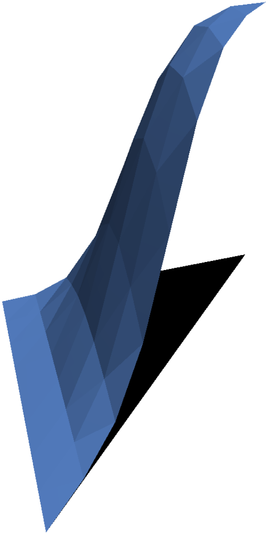
Numerical Experiments

NODAL BASIS FUNCTION: REFINEMENT LEVEL 1

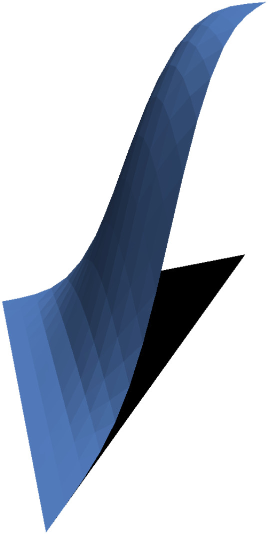
NODAL BASIS FUNCTION: REFINEMENT LEVEL 2



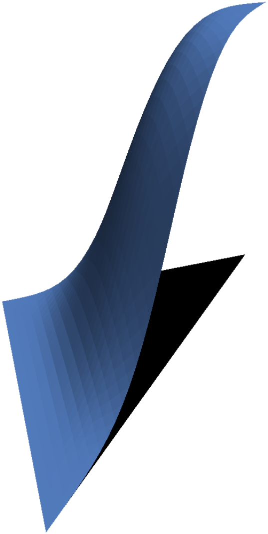
NODAL BASIS FUNCTION: REFINEMENT LEVEL 3



NODAL BASIS FUNCTION: REFINEMENT LEVEL 4

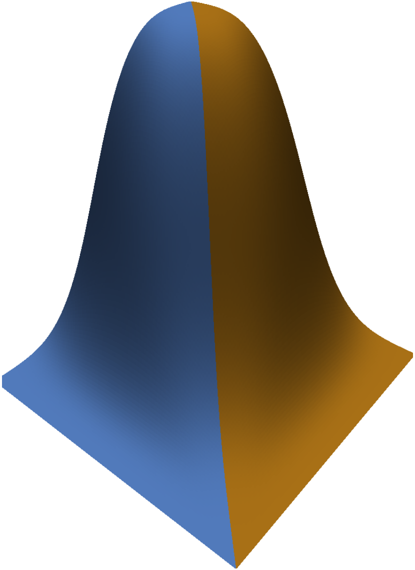


NODAL BASIS FUNCTION: REFINEMENT LEVEL 5



NODAL BASIS FUNCTION: REFINEMENT LEVEL 6

SMOOTHNESS: PLOTTING DERIVATIVES



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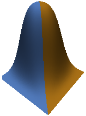

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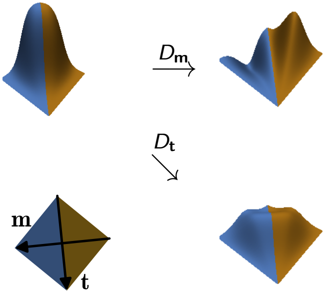

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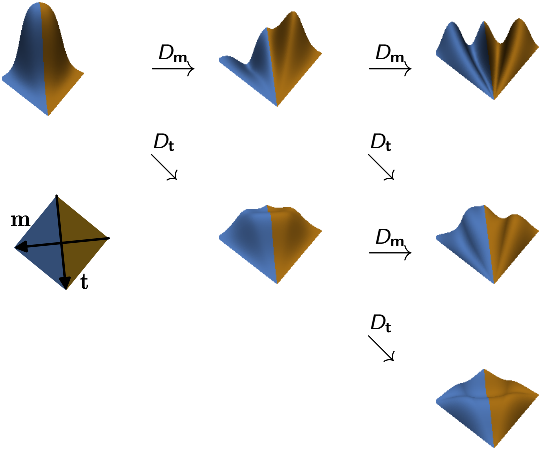

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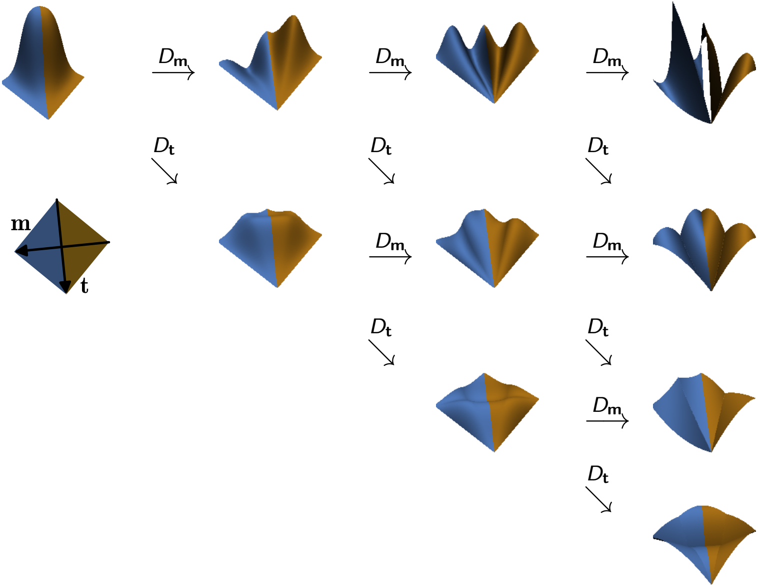
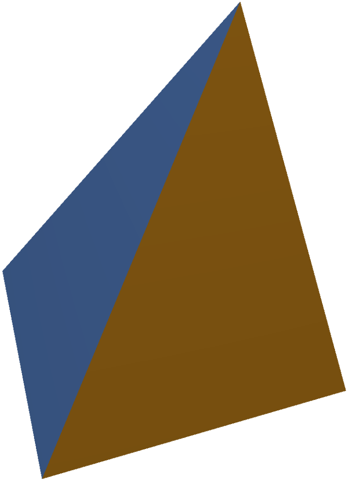
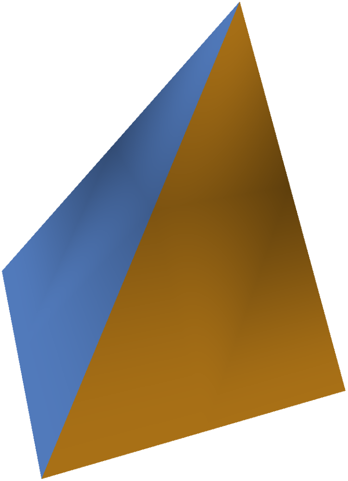


FLAT AND PHONG SHADING: LEVEL 1

Flat shading



Phong shading



FLAT AND PHONG SHADING: LEVEL 2

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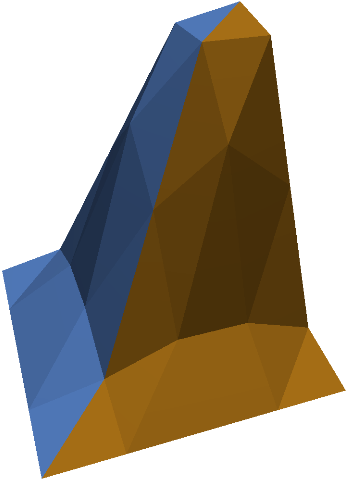

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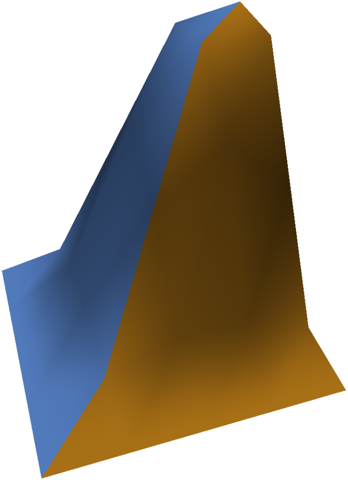
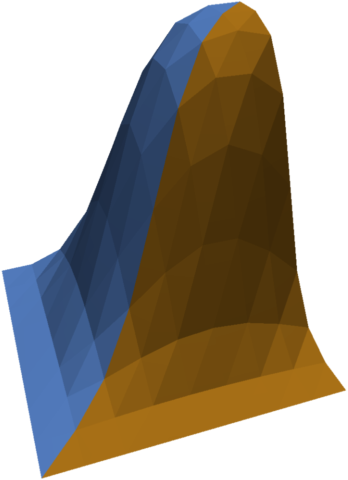


FLAT AND PHONG SHADING: LEVEL 3

Flat shading



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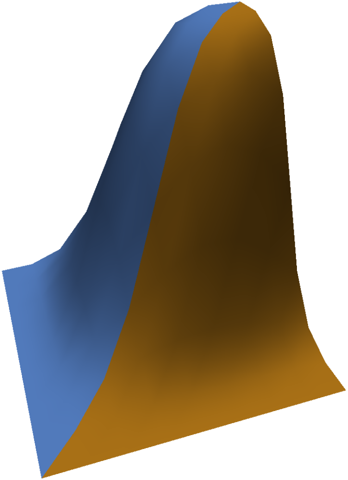
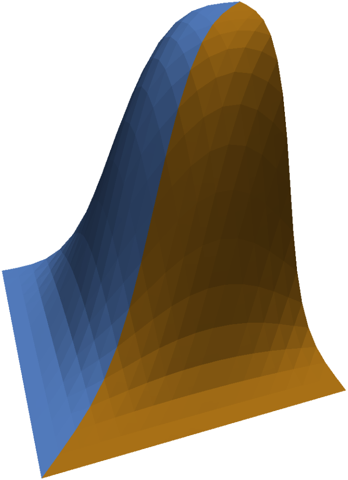


FLAT AND PHONG SHADING: LEVEL 4

Flat shading



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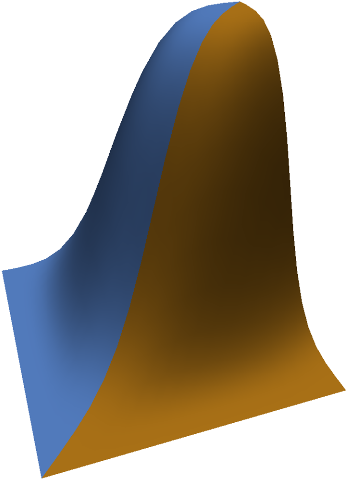
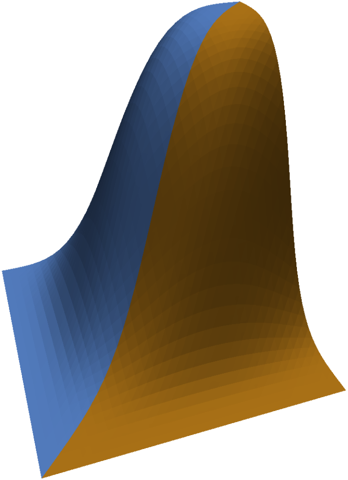


FLAT AND PHONG SHADING: LEVEL 5

Flat shading



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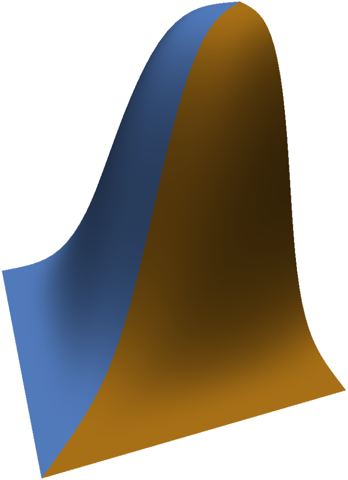
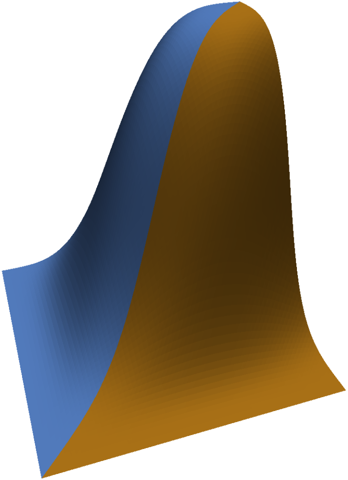


FLAT AND PHONG SHADING: LEVEL 6

Flat shading



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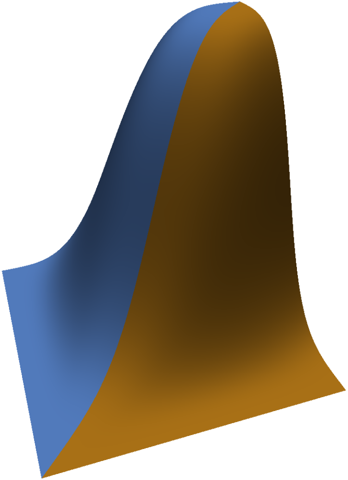


Thank you!

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References II

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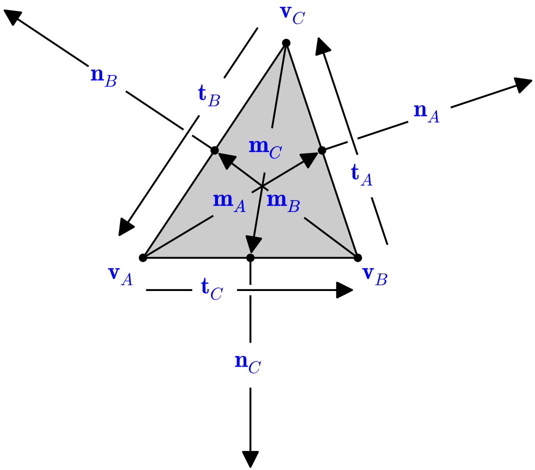
Appendix

BASIS FOR DIRECTIONAL DERIVATIVES



$$\mathbf{t}_A + \mathbf{t}_B + \mathbf{t}_C = 0$$

$$\mathbf{n}_A + \mathbf{n}_B + \mathbf{n}_C = 0$$

$$\mathbf{m}_A + \mathbf{m}_B + \mathbf{m}_C = 0$$

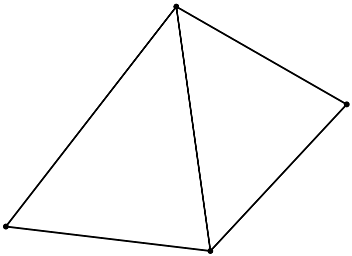
$$\mathbf{m}_A = \frac{1}{2}(\mathbf{t}_B - \mathbf{t}_C)$$

$$\mathbf{t}_A = \frac{2}{3}(\mathbf{m}_C - \mathbf{m}_B)$$

THE POWELL-SABIN SPLITS

THE POWELL-SABIN SPLITS

6-split



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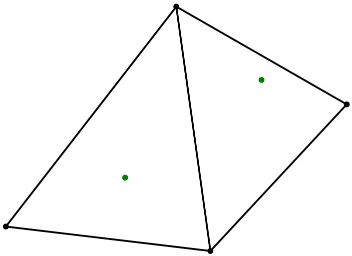
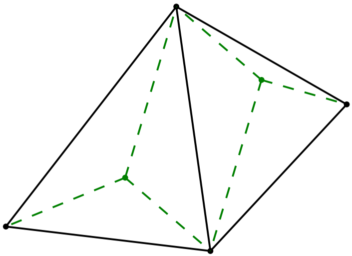


THE POWELL-SABIN SPLITS

6-split



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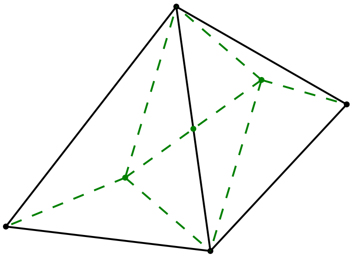

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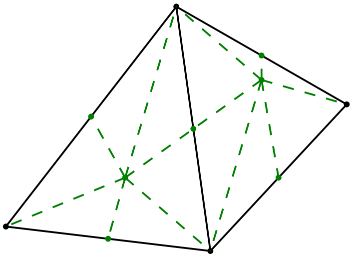

THE POWELL-SABIN SPLITS

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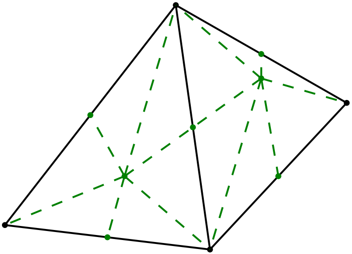
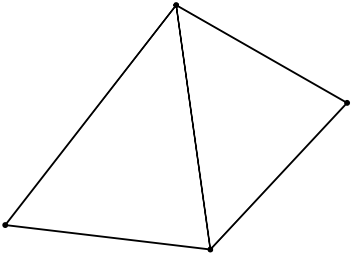


12-split



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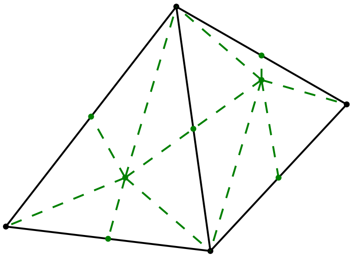

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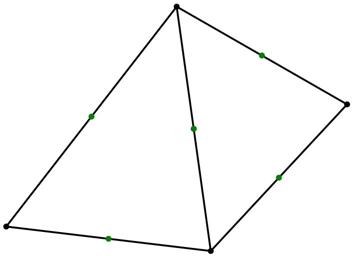

THE POWELL-SABIN SPLITS

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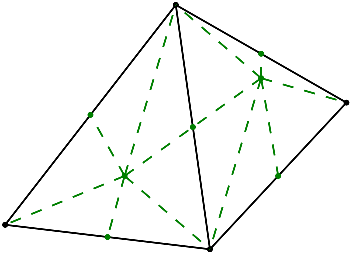

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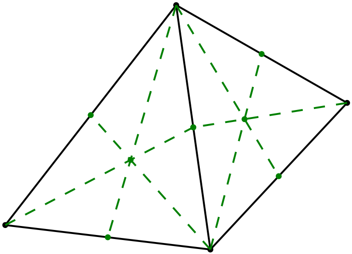

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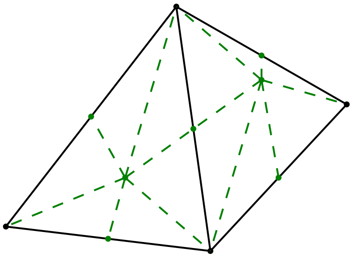

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