MAIA conference
Erice (Italy), September 26, 2013

On multivariate Multi-Resolution Analysis, using generalized (non homogeneous) polyharmonic splines

or: A way for deriving RBF and associated MRA

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1. Some known tools... using Fourier Transform

2. Extension of polyharmonic splines, and associated MRA

Part 1:

Some known tools...

using Fourier Transform

A word on (odd degree) polynomial splines

Definition:

$$\sigma_m \;=\; \texttt{Argmin}_{\forall i\in[1:n],\, f(x_i)=y_i} \int_{\mathbb{R}} (f^{(m)}(x))^2\, dx$$

$$\Longrightarrow\; \sigma_m(x) = \sum_{i=1:n} \lambda_i\, |x - x_i|^{2m-1} + p_{m-1}(x)$$

with $\forall q \in \mathbb{P}_{m-1}\,,\ \sum_{i=1:n} \lambda_i q(x_i) = 0$ and $p_{m-1} \in \mathbb{P}_{m-1}$

"Radial basis functions": writing $u_m(x) = \dfrac{1}{2(m!)}\,|x|^{2m-1}$

$$\sigma_m(x) = \sum_{i=1:n} \mu_i\, u_m(x - x_i) + q_{m-1}(x)$$

(same constraints on μ and q_{m-1})

Derivative and Fourier Transform:

$$E(u_m) \;:=\; (-1)^m\, u_m^{(2m)} = Dirac$$

$$\widehat{u_m}(\omega) \;=\; \frac{1}{\omega^{2m}}$$

... and on associated functions

"Basic spline", or "B-spline": $\varphi_m = (-1)^m \delta^{2m} u_m$ $\qquad \widehat{\varphi_m}(\omega) = \dfrac{|\sin\omega|^{2m}}{|\omega|^{2m}}$

"Lagrangian spline", or "L-spline": $L_m(0) = 1\ ; \quad \forall j \in \mathbb{Z}\backslash\{0\}, L_m(j) = 0$

$$L_m = \sum_{j\in\mathbb{Z}} L_m(j/2)\, L_m(2\bullet - j) \quad ; \quad \varphi_m = \sum_{j\in\mathbb{Z}} \varphi_m(j)\, L_m(\bullet - j)$$

$$\widehat{L_m}(\omega) = \frac{\widehat{\varphi_m}(\omega)}{\sum_{\ell\in\mathbb{Z}} \widehat{\varphi_m}(\omega - 2\pi\ell)} = \frac{\widehat{\varphi_m}(\omega)}{\sum_{j\in\mathbb{Z}} \varphi_m(j) \exp(-i\,j\,\omega)} = \frac{\omega^{-2m}}{\sum_{\ell\in\mathbb{Z}} (\omega - 2\pi\ell)^{-2m}}$$

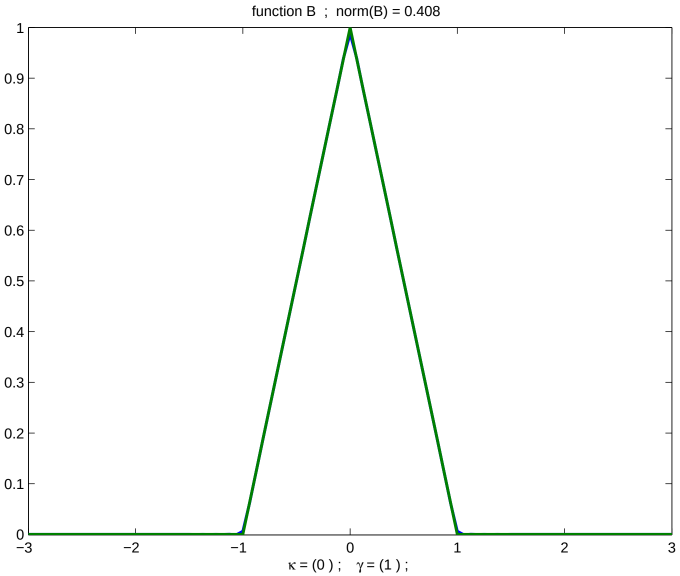
Semi-orthogonal wavelet: $\psi_m = (-1)^m D^{2m} L_{2m}$ (to be normalized)

$$\widehat{\psi_m}(\omega) = \omega^{2m} \widehat{L_{2m}}(\omega) = \frac{\omega^{-2m}}{\sum_{\ell\in\mathbb{Z}} (\omega - 2\pi\ell)^{-4m}}$$

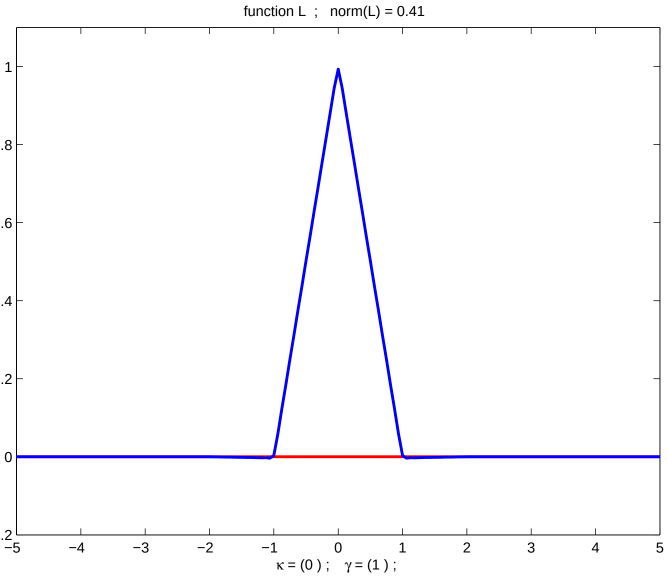
Orthogonal wavelet: $\widehat{\psi^{\perp}}(\omega) = \dfrac{\widehat{\psi}(\omega)}{\sqrt{\sum_{\ell\in\mathbb{Z}} (\widehat{\psi}(|\omega - 2\pi\ell|))^2}} = \dfrac{\omega^{-2m}}{\sqrt{\sum_{\ell\in\mathbb{Z}} (\omega - 2\pi\ell)^{-4m}}}$

Linear case

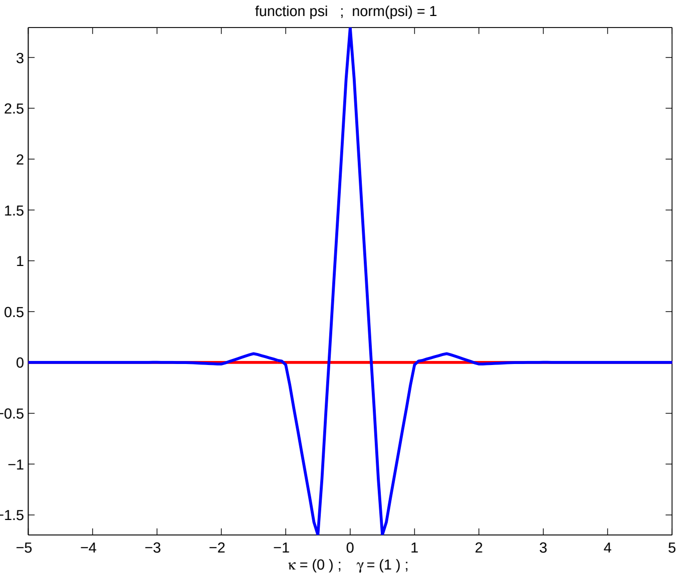
B-spline



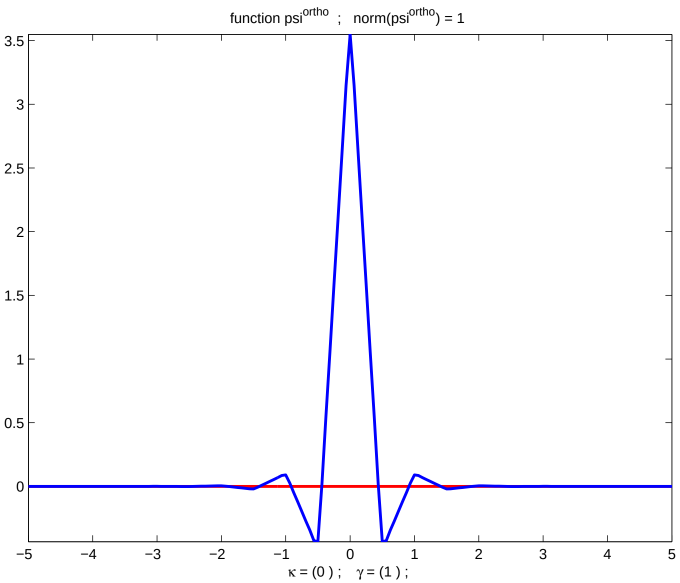
L-spline



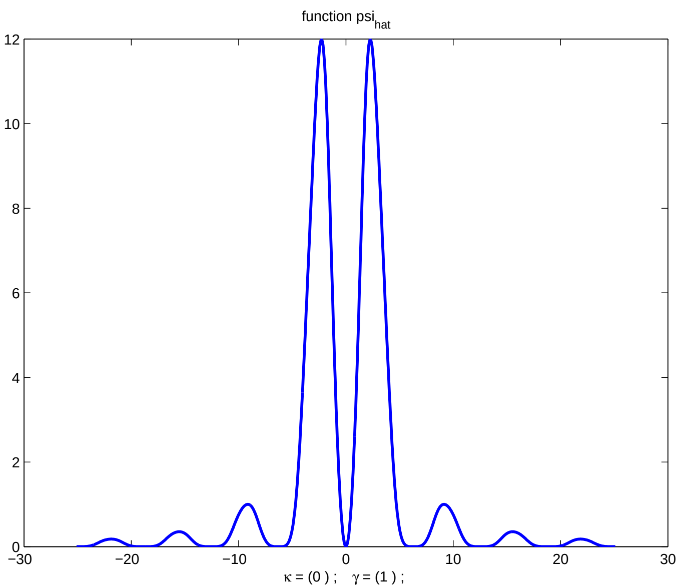
psi-spline



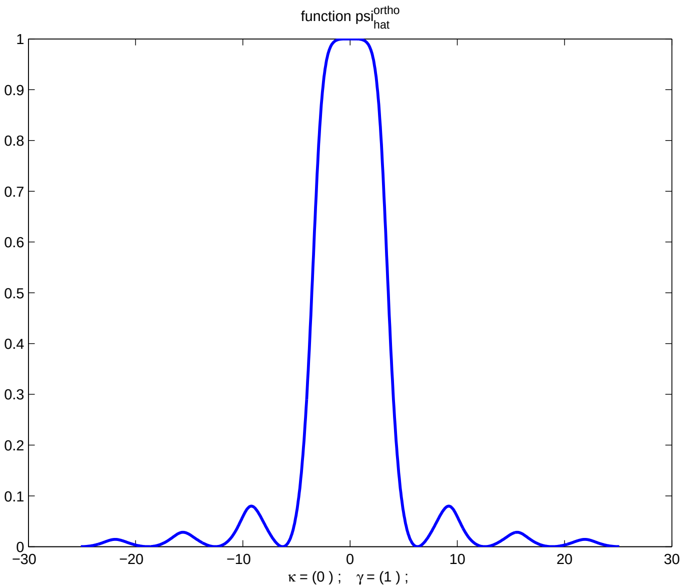
psi-ortho



psi-hat

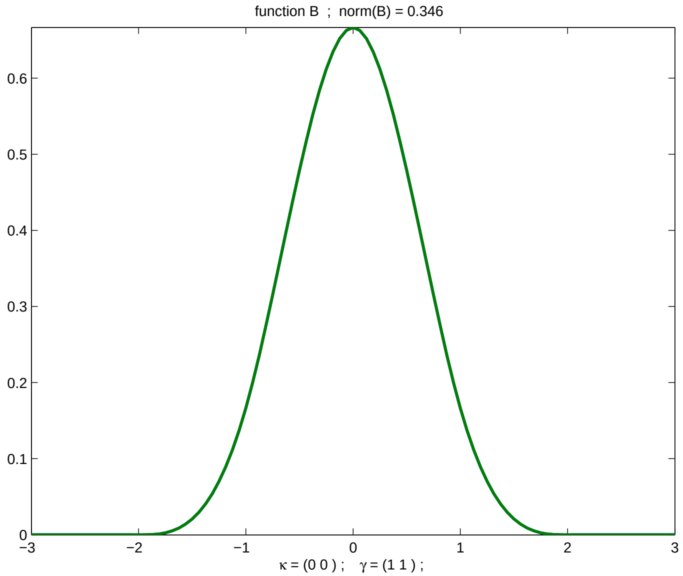


psi-ortho-hat

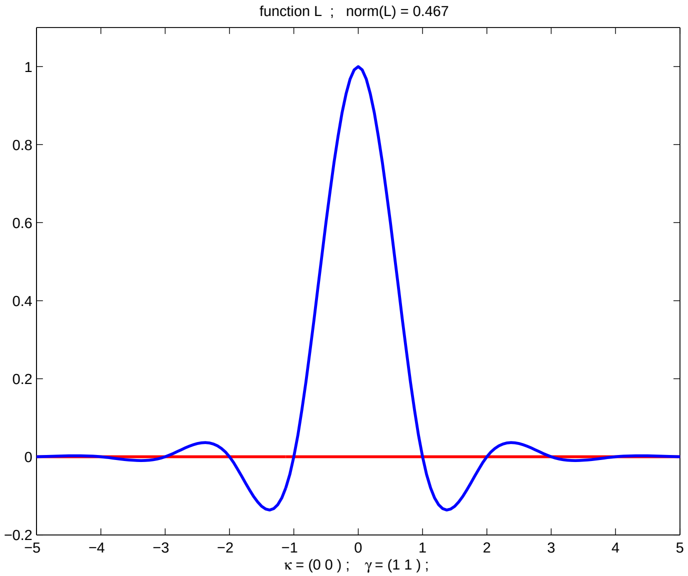


Cubic case

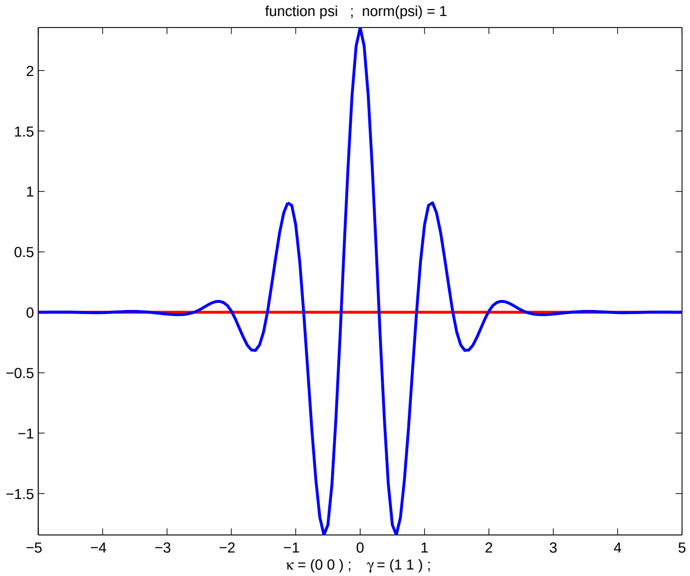
B-spline



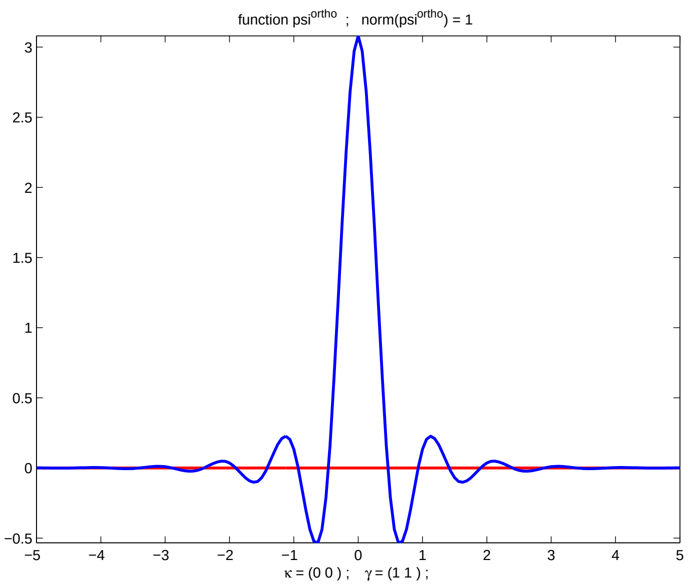
L-spline



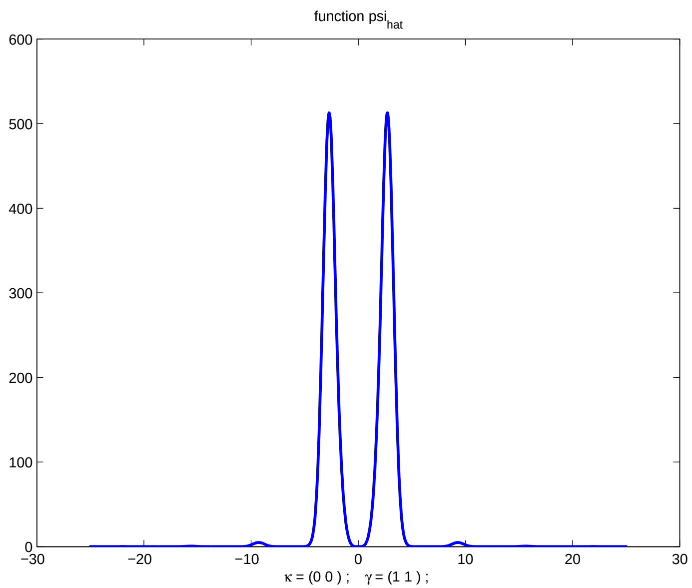
psi-spline



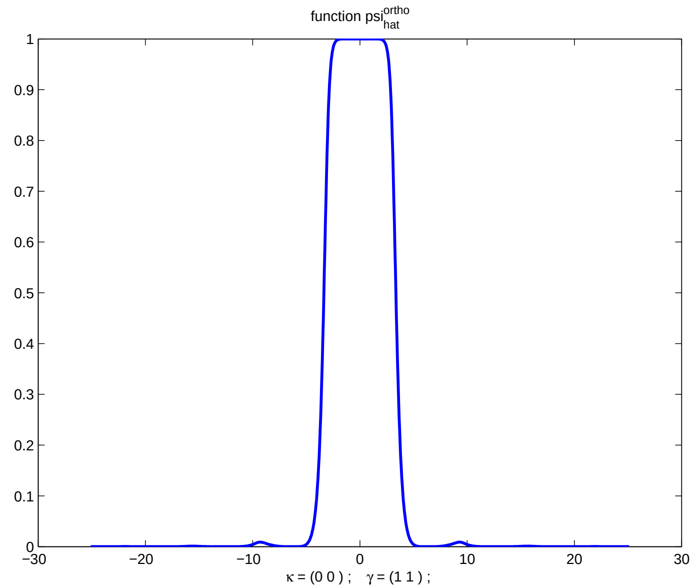
psi-ortho



psi-hat



psi-ortho-hat



Use of the associated functions
(translation invariant spaces)

$\widehat{y}$ is defined by $\widehat{y}(\omega) = \sum_{j\in\mathbb{Z}} y_j \exp(-i\,j\,\omega)$.

B-spline approximation of vector y : (or of points P_j for B-spline curve)

$$\sigma_m = \sum_{j\in\mathbb{Z}} y_j\,\varphi_m(\bullet - j) \qquad \Longleftrightarrow \qquad \widehat{\sigma} = \widehat{y}\,\widehat{\varphi}_m.$$

Interpolating spline of vector y : (or P_j instead of y_j for interpolating spline curve)

$$\sigma_m = \sum_{j\in\mathbb{Z}} y_j\,L_m(\bullet - j)$$

$$\widehat{\sigma} = \widehat{y}\,\widehat{L}.$$

Wavelet decomposition of some $f \in L^2(\mathbb{R})$:

$$f = \sum_{j\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}} (f\,,\psi^{\perp}(2^{\ell}\bullet - j))\,\psi^{\perp}(2^{\ell}\bullet - j)$$

$$(f\,,\psi^{\perp}(2^{\ell}\bullet - j)) = 2^{-\ell}\exp(i\,2^{-\ell}j)\,(\widehat{f}\,,\widehat{\psi^{\perp}}(2^{-\ell}\bullet)$$

Spline under tension

Definition and Fourier transform

$$\sigma_{m,k} = \mathtt{Argmin}_{\forall i \in \mathbb{Z},\, f(x_i)=y_i} \int_{\mathbb{R}} (f^{(m)}(x))^2 dx + \rho^2 \int_{\mathbb{R}} (f^{(k)}(x))^2\, dx \qquad (\text{say } k < m)$$

$$E_\rho(u) := (-1)^m\, D^{2m} u + (-1)^k\, \rho^2\, D^{2k} u = Dirac \qquad ; \qquad \widehat{u_\rho}(\omega) = \frac{1}{\omega^{2m} + \rho^2\, \omega^{2k}}$$

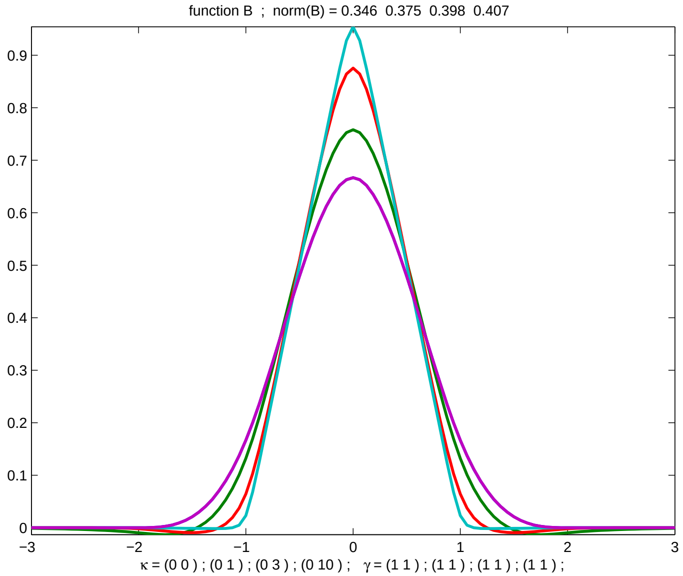
$$\widehat{\varphi_\rho}(\omega) = \frac{\sin^{2m}\omega + \rho^2\, \sin^{2k}\omega}{\omega^{2m} + \rho^2\, \omega^{2k}} \qquad \widehat{L_\rho}(\omega) = \frac{\left(\omega^{2m} + \rho^2\, \omega^{2k}\right)^{-1}}{\sum_{\ell\in\mathbb{Z}} \left((\omega - 2\pi\ell)^{2m} + \rho^2\, (\omega - 2\pi\ell)^{2k}\right)^{-1}}$$

$$\widehat{\psi_\rho}(\omega) = \omega^{2m}\widehat{L_\rho^2}(\omega) + \rho^2\omega^{2k}\widehat{L_\rho^2}(\omega) = \frac{\left(\omega^{2m} + \rho^2\, \omega^{2k}\right)^{-1}}{\sum_{\ell\in\mathbb{Z}} \left((\omega - 2\pi\ell)^{2m} + \rho^2(\omega - 2\pi\ell)^{2k}\right)^{-2}}$$

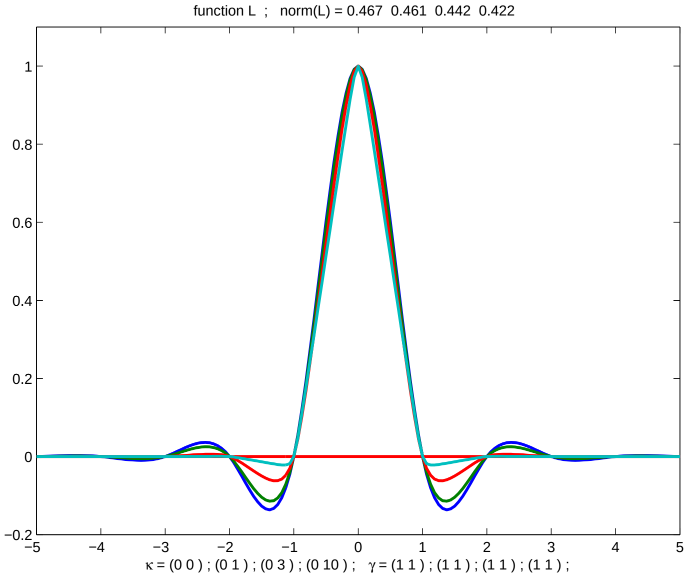
$$\widehat{\psi_\rho^{\perp}}(\omega) = \frac{\left(\omega^{2m} + \rho^2\, \omega^{2k}\right)^{-1}}{\sqrt{\sum_{\ell\in\mathbb{Z}} \left((\omega - 2\pi\ell)^{2m} + \rho^2(\omega - 2\pi\ell)^{2k}\right)^{-2}}}$$

Tension splines (linear-cubic)

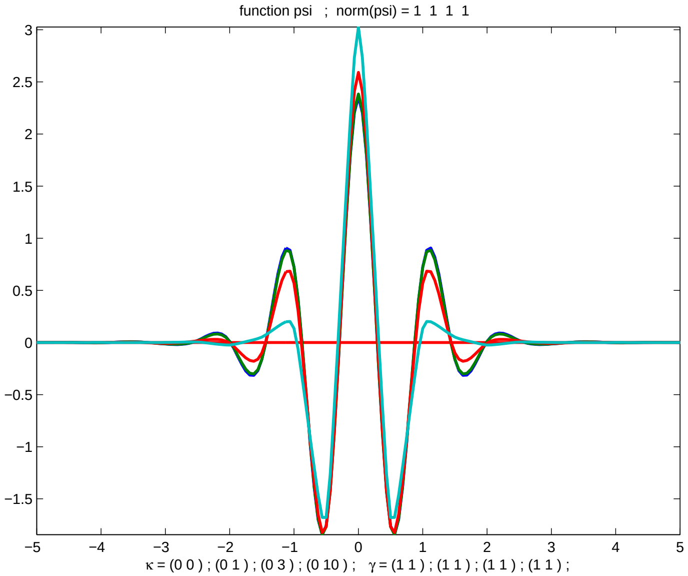
B-spline



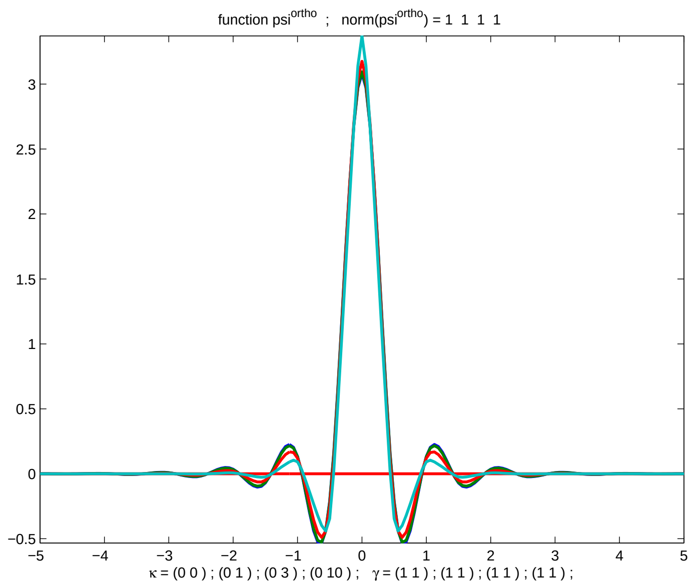
L-spline



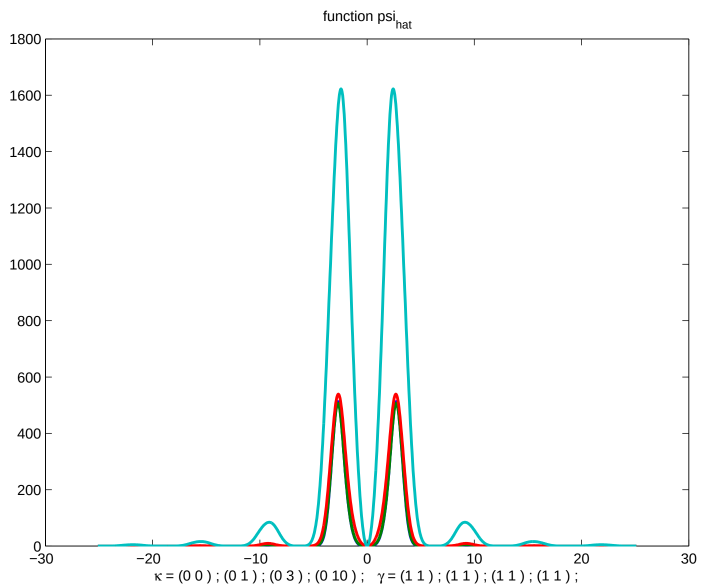
psi-spline



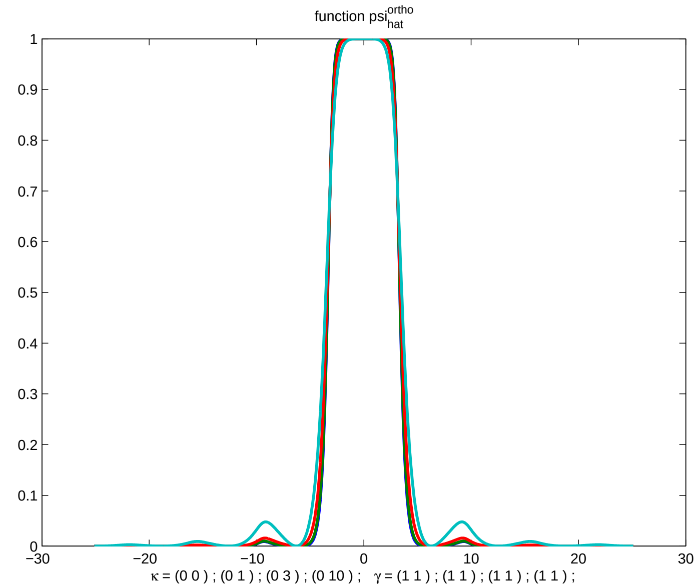
psi-ortho



psi-hat

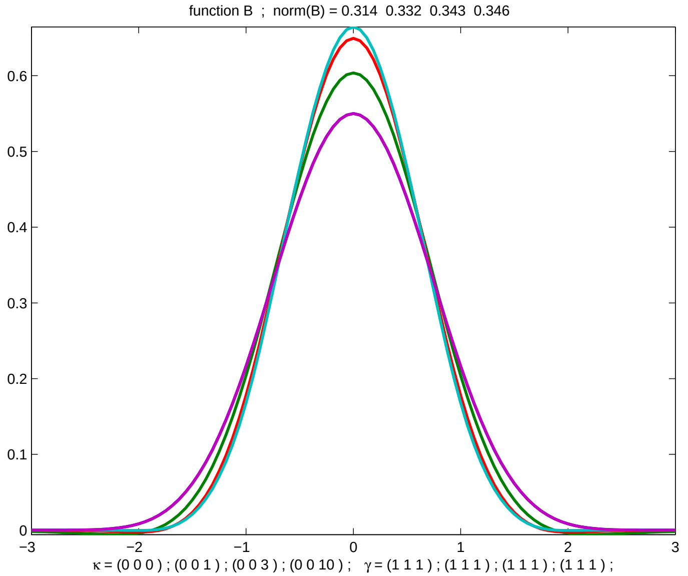


psi-ortho-hat

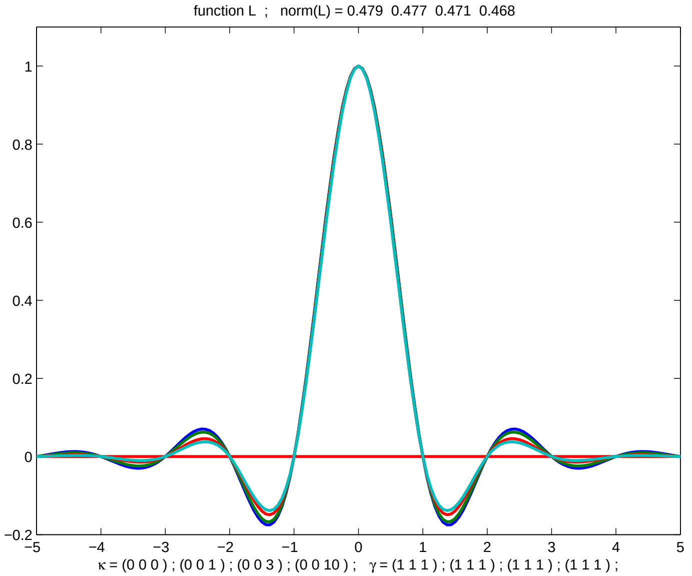


Tension splines (cubic-quintic)

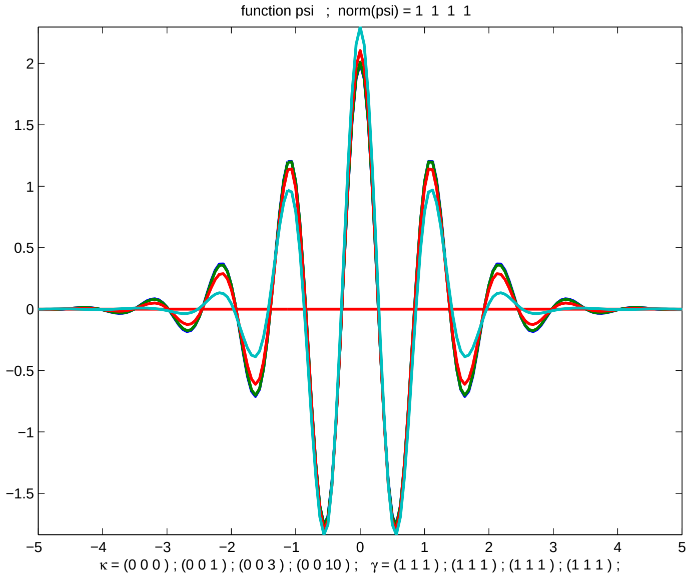
B-spline



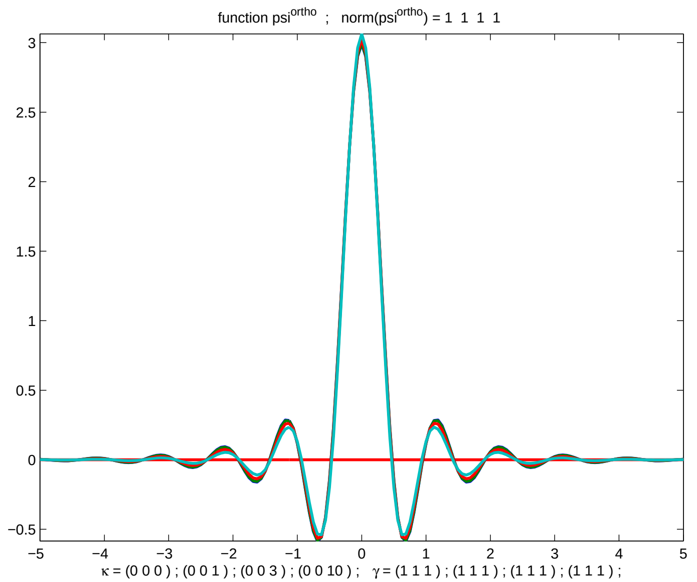
L-spline



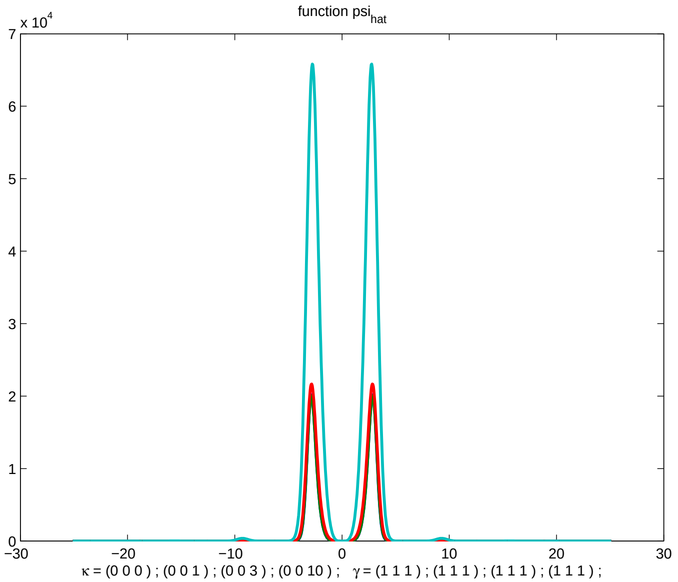
psi-spline



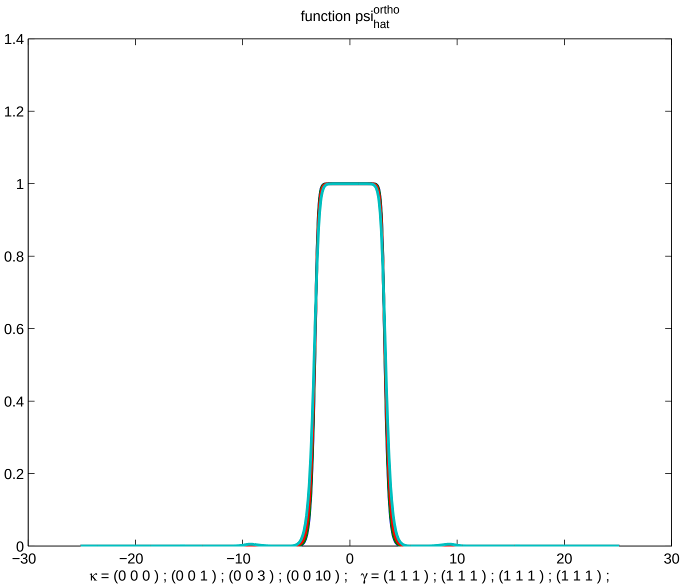
psi-ortho



psi-hat



psi-ortho-hat



Fractional splines

Definition and Fourier transform

Let $s \in [0..1)$ and $m \in \mathbb{N}$ such that $\alpha := m + s > d/2$

$$\sigma_\alpha = \texttt{Argmin}_{\forall i \in [1:n],\, f(x_i)=y_i} \int_{\mathbb{R}} (f^{(\alpha)}(x))^2 dx$$
$$= \texttt{Argmin}_{\forall i \in [1:n],\, f(x_i)=y_i} \int_{\mathbb{R}} (\omega^2)^s \left(\mathcal{F}(f^{(m)}(\omega)\right)^2 d\omega$$

$$E_\alpha(u_\alpha) := (-1)^{\lfloor \alpha \rfloor} D^{2\alpha} u_\alpha = Dirac \quad ; \quad \widehat{u}_\alpha(\omega) = \frac{1}{(\omega^2)^\alpha}$$

$u_\alpha(x) = c_\alpha |x|^{2\alpha-1}$ if $2\alpha - 1$ is not an even integer number.

$u_\alpha(x) = c_\alpha |x|^{2\alpha-1} \ln x^2$ if $2\alpha - 1$ is an even integer number.

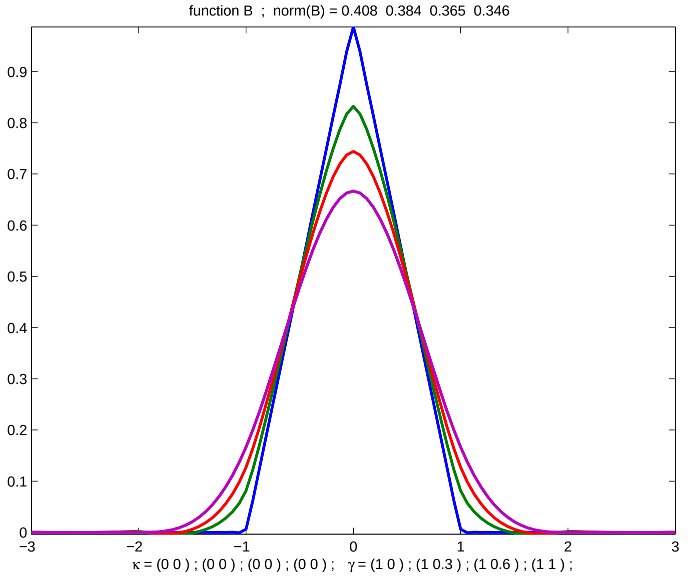
(c_α is some known real valued constant)

They too are in some place between order 1 (linear) and order 2 (cubic) splines, but are different from splines under tension.

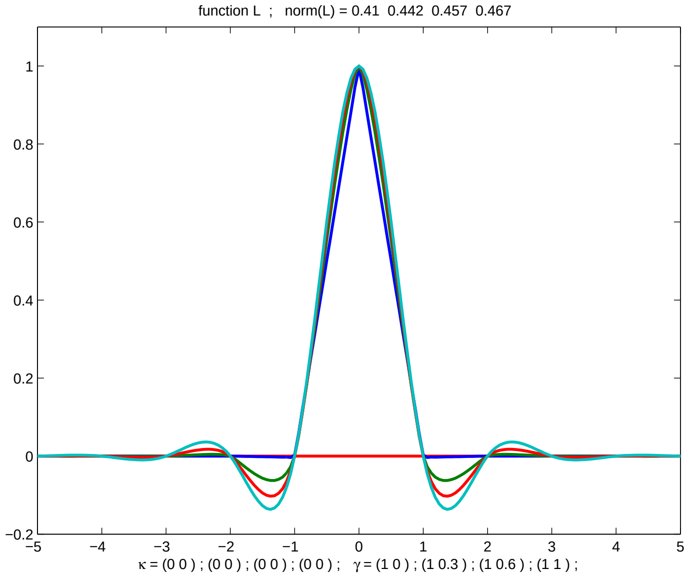
B-spline : $\widehat{\varphi}_\alpha(\omega) = \left(\frac{\sin^2 \omega}{\omega^2}\right)^\alpha$

Fractional splines (linear-cubic)

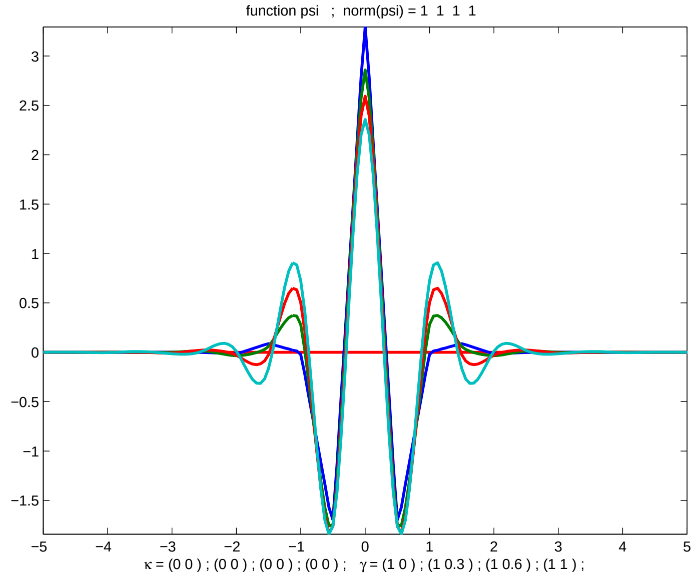
B-spline



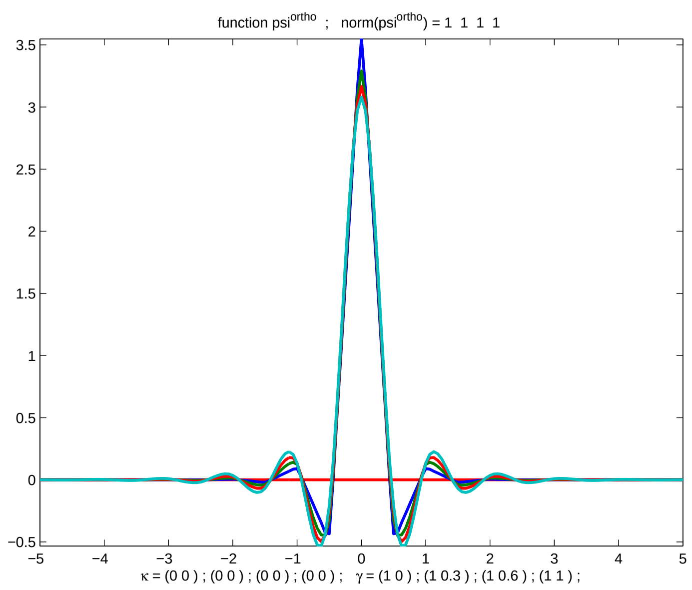
L-spline



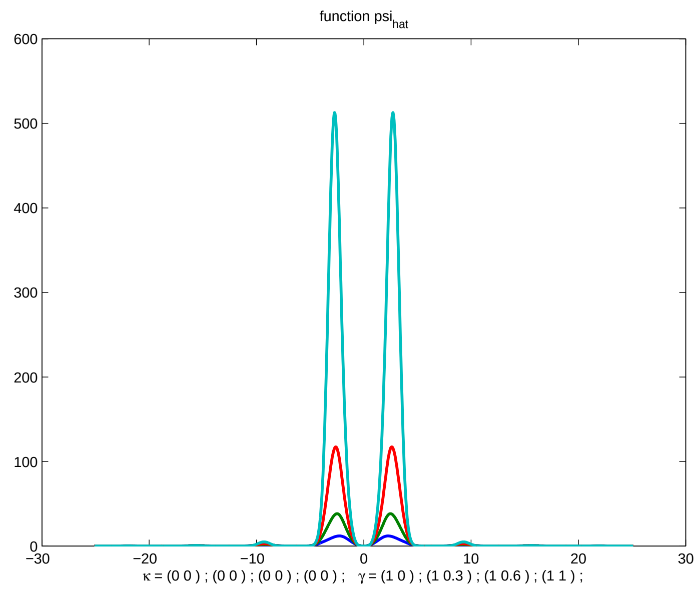
psi-spline



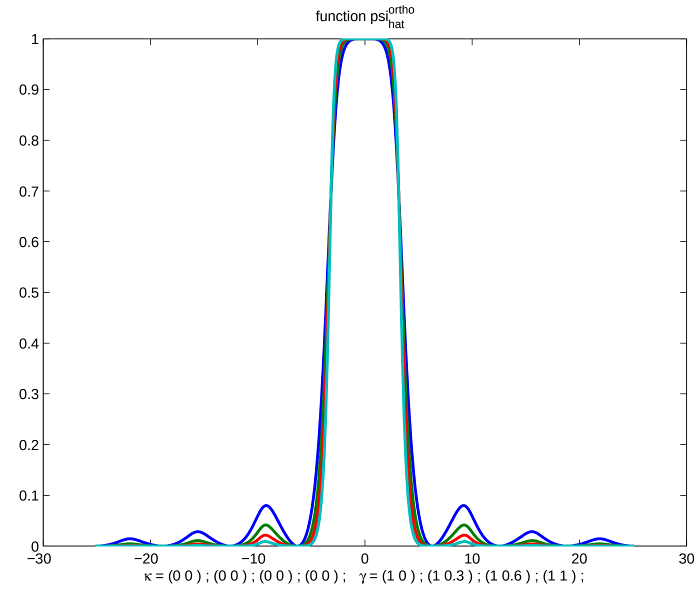
psi-ortho



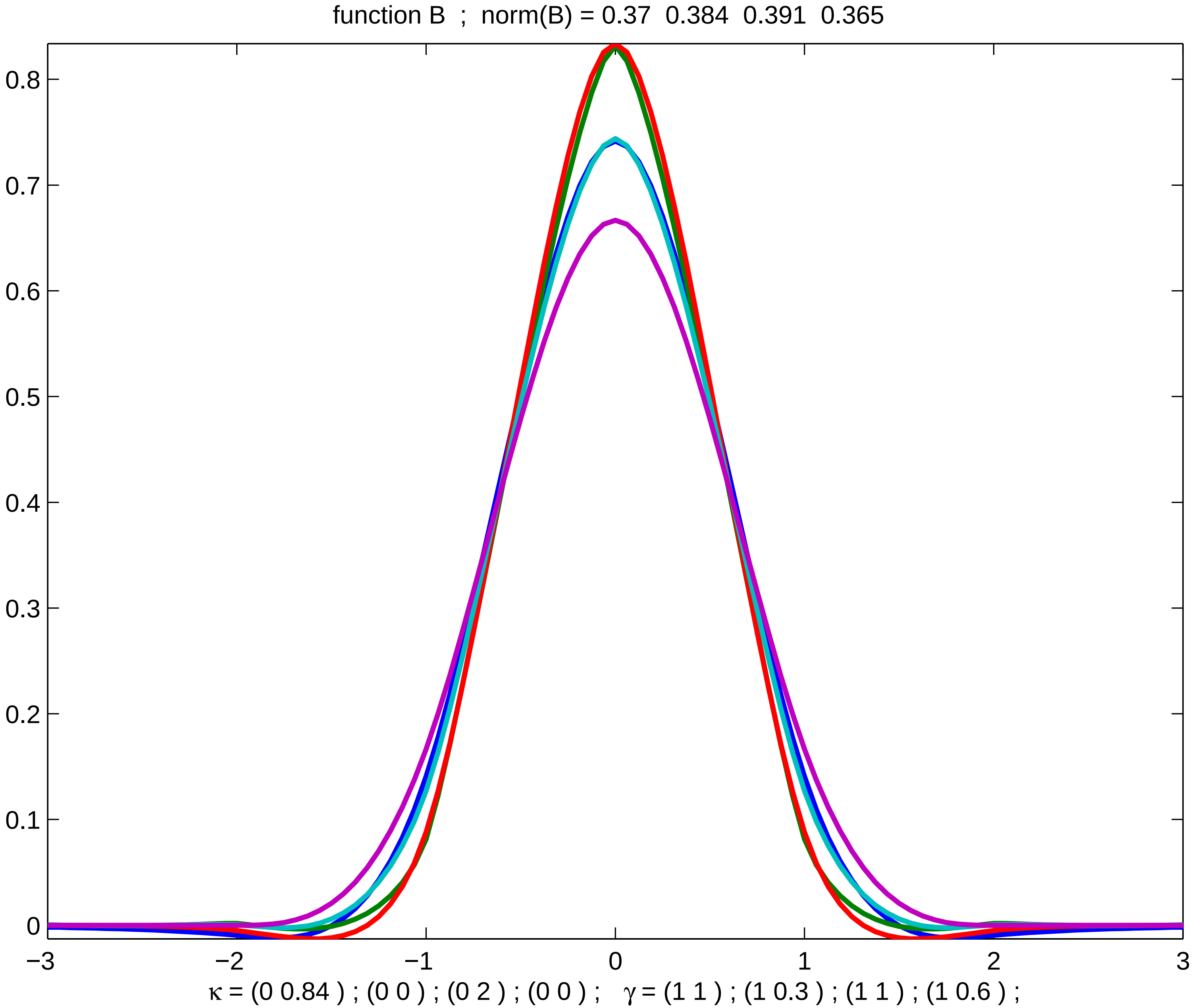
psi-hat



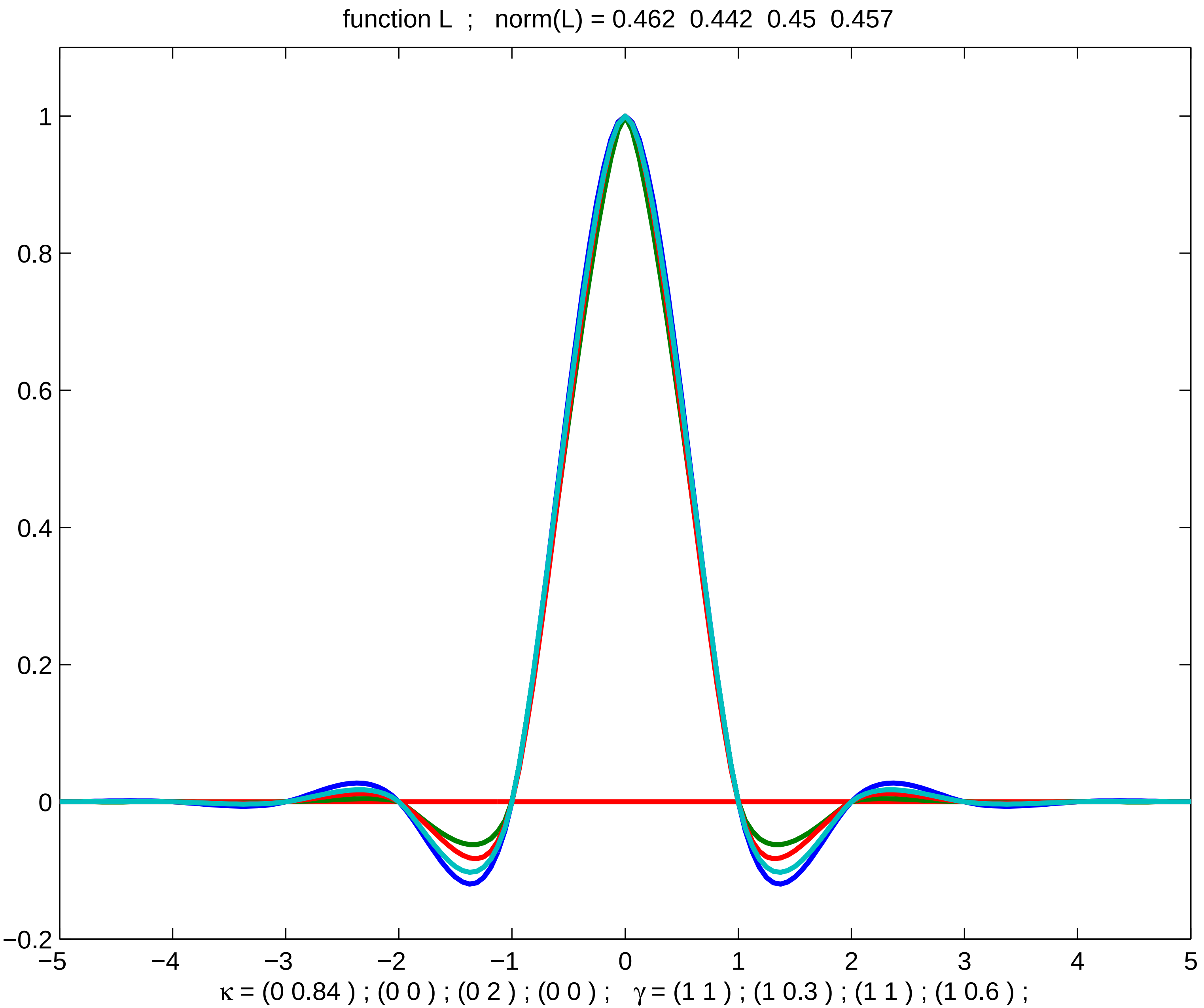
psi-ortho-hat



comparison tension B-splines versus fractional B-splines



comparison tension L-splines versus fractional L-splines



(d-dimensional, real order) polyharmonic splines

Definition and expression

Let $s \in [0..1)$ and $m \in \mathbb{N}$ such that $\alpha := m + s > 1/2$

$$\sigma_\alpha = \texttt{Argmin}_{\forall i \in [1:n],\, f(x_i)=y_i} \int_{\mathbb{R}^d} \|D^\alpha f(x)\|^2 \, dx$$

$$= \texttt{Argmin}_{\forall i \in [1:n],\, f(x_i)=y_i} \int_{\mathbb{R}^d} \|\omega\|^{2s} \, \|\mathcal{F}(D^m f)(\omega)\|^2 \, dx$$

$$\Longrightarrow \sigma_\alpha(x) = \sum_{i=1:n} \lambda_i \, u_\alpha(x - x_i) + p_{m-1}(x)$$

$$\text{with } \forall q \in \mathbb{P}_{m-1} \, , \; \sum_{i=1:n} \lambda_i q(x_i) = 0 \text{ and } p_{m-1} \in \mathbb{P}_{m-1}$$

where $E_\alpha(u_\alpha) := (-1)^m \, \Delta^\alpha u_\alpha = Dirac$

$u_\alpha(x) = c_\alpha \, \|x\|^{2\alpha-d}$ if $2\alpha - d$ is not an even integer number.

$= c_\alpha \, \|x\|^{2\alpha-d} \ln \|x\|^2$ if $2\alpha - d$ is an even integer number.

Radial basis functions (no other known writing)

(d-dimensional, real order) polyharmonic splines

Derivative and Fourier Transform $\Delta^\alpha u_\alpha = Dirac$ $\widehat{u_\alpha}(\omega) \;=\; \frac{1}{\|\omega\|^{2\alpha}}$

Polyharmonic B-spline: $\varphi_\alpha = \Delta^\alpha u_\alpha$ $\widehat{\varphi_\alpha}(\omega) \;=\; \left(\frac{\|\sin\omega\|^2}{\|\omega\|^2}\right)^\alpha$

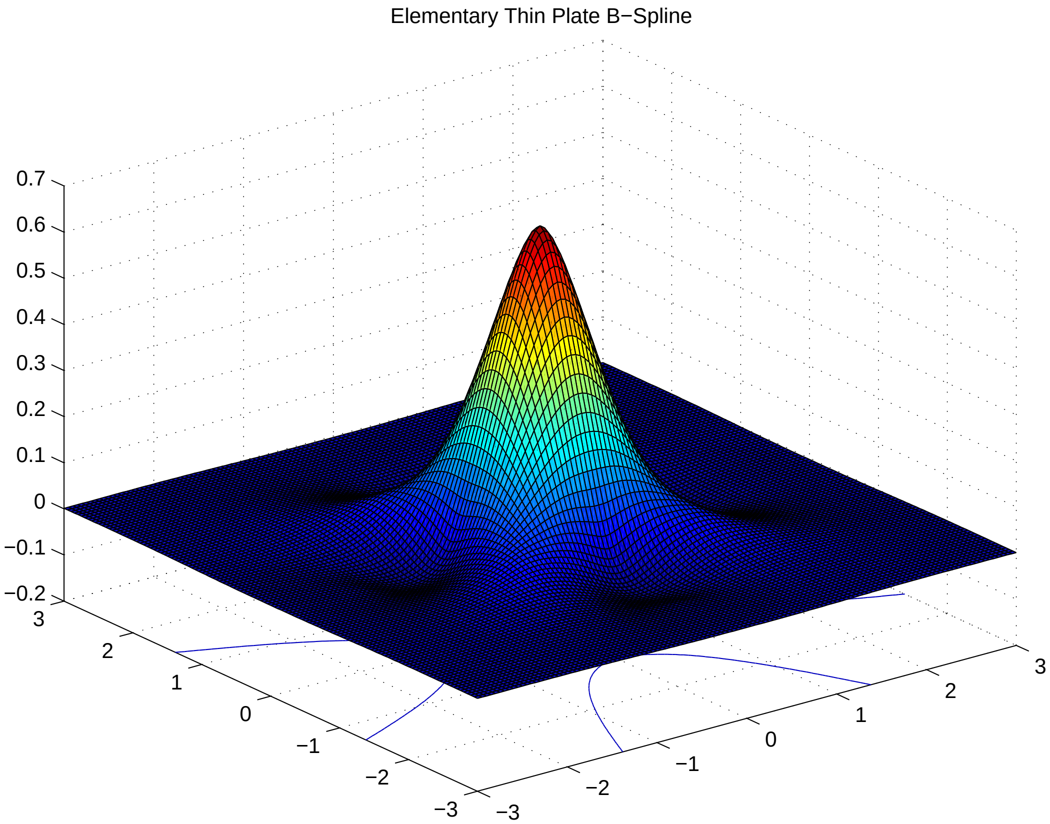
"Lagrangian spline", or "L-spline": $\widehat{L_\alpha}(\omega) \;=\; \frac{\|\omega\|^{-2\alpha}}{\sum_{\ell\in\mathbb{Z}^d}(\|\omega-2\pi\ell\|)^{-2\alpha}}$

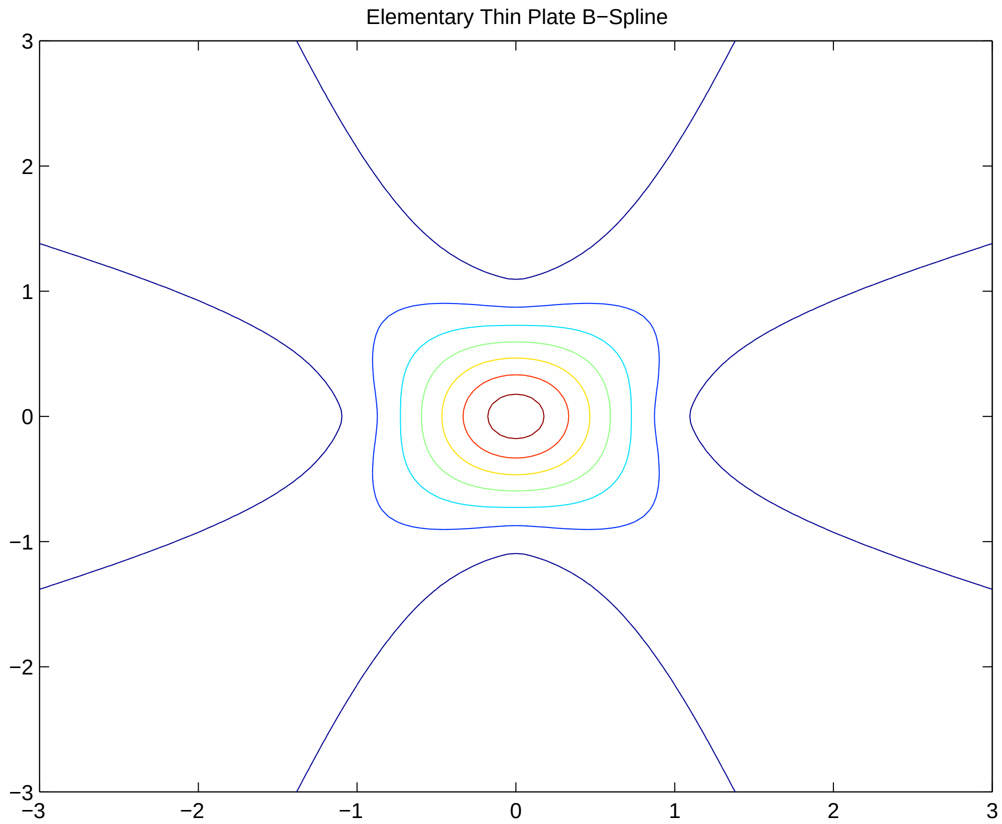
Wavelets:

$$\psi_\alpha \;=\; \Delta^\alpha L_{2\alpha} \qquad \widehat{\psi_\alpha}(\omega) \;=\; \left(\|\omega\|^2\right)^\alpha \widehat{L_{2\alpha}}(\omega) \;=\; \frac{\left(\|\omega\|^2\right)^{-\alpha}}{\sum_{\ell\in\mathbb{Z}^d}\left(\|\omega-2\pi\ell\|^2\right)^{-2\alpha}}$$

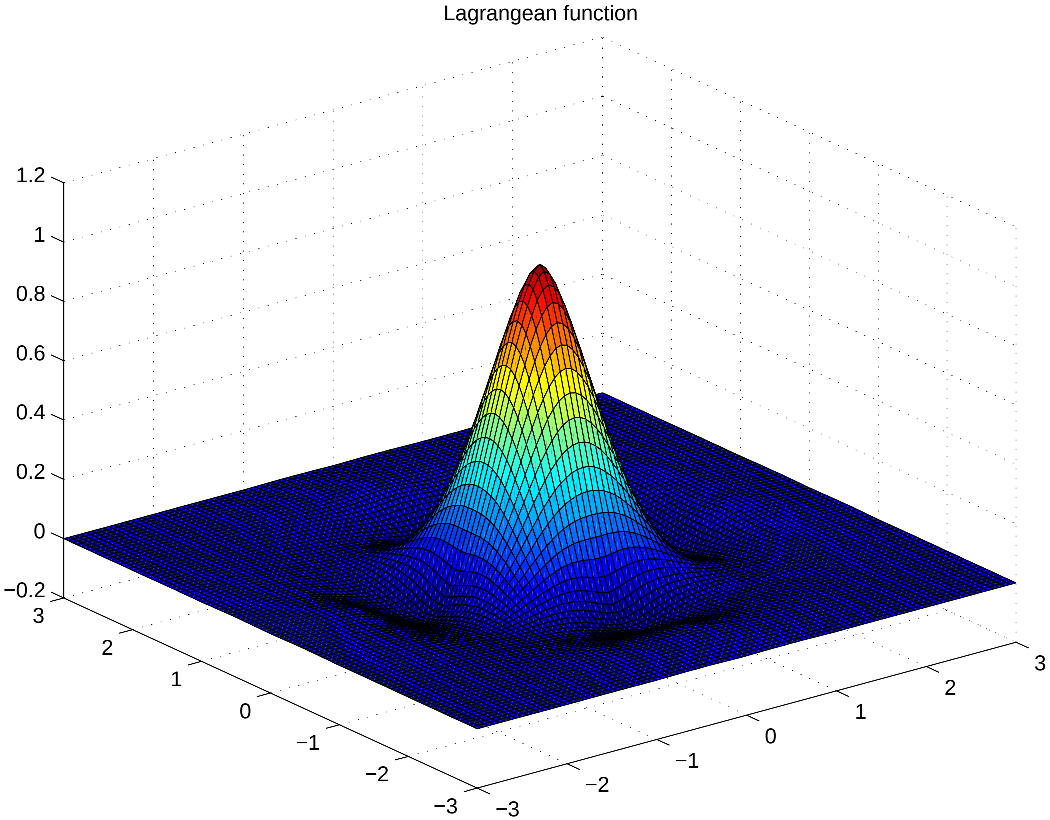
$$\psi_\alpha^\perp(\omega) \;=\; \frac{\left(\|\omega\|^2\right)^{-\alpha}}{\sqrt{\sum_{\ell\in\mathbb{Z}^d}((\|\omega-2\pi\ell\|^2)^{-2\alpha}}}$$

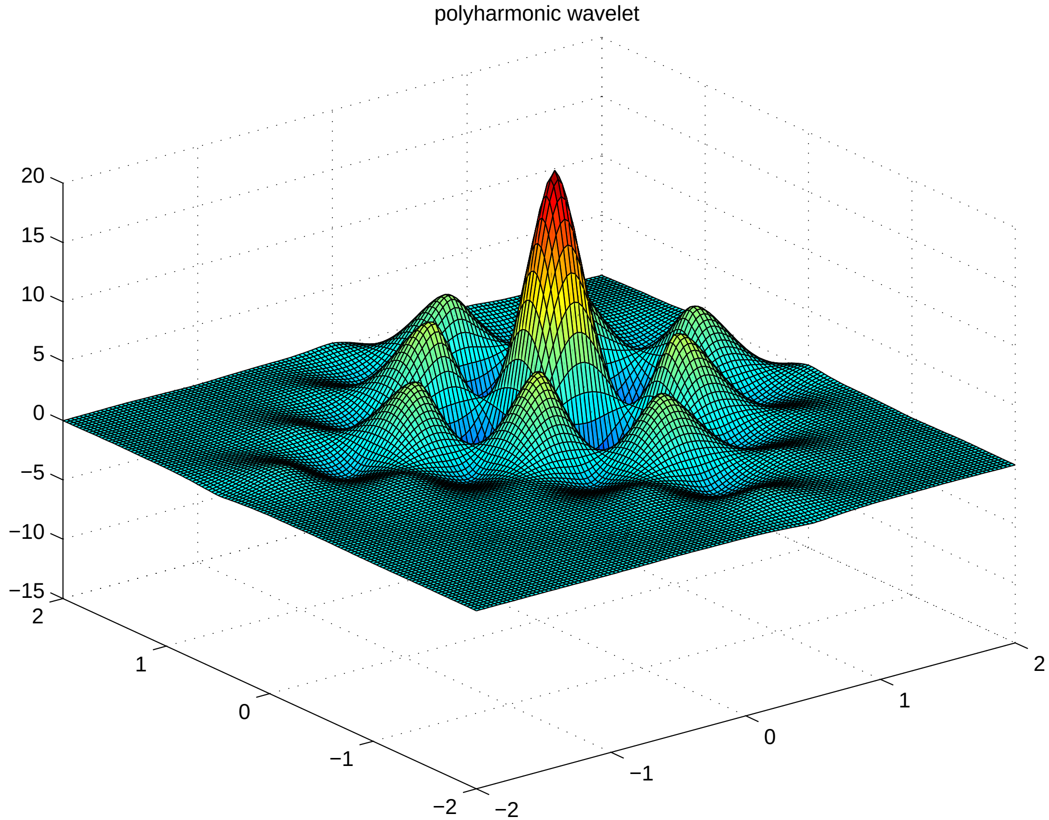
Biharmonic (“thin plate”) B-spline



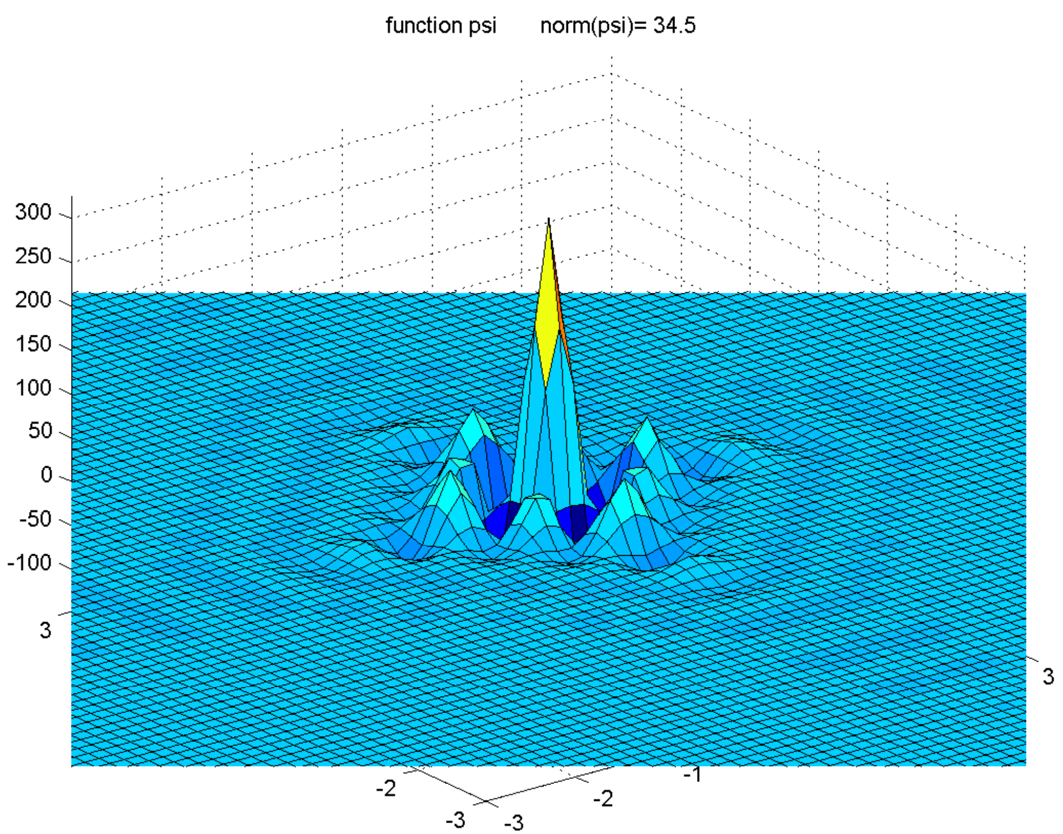


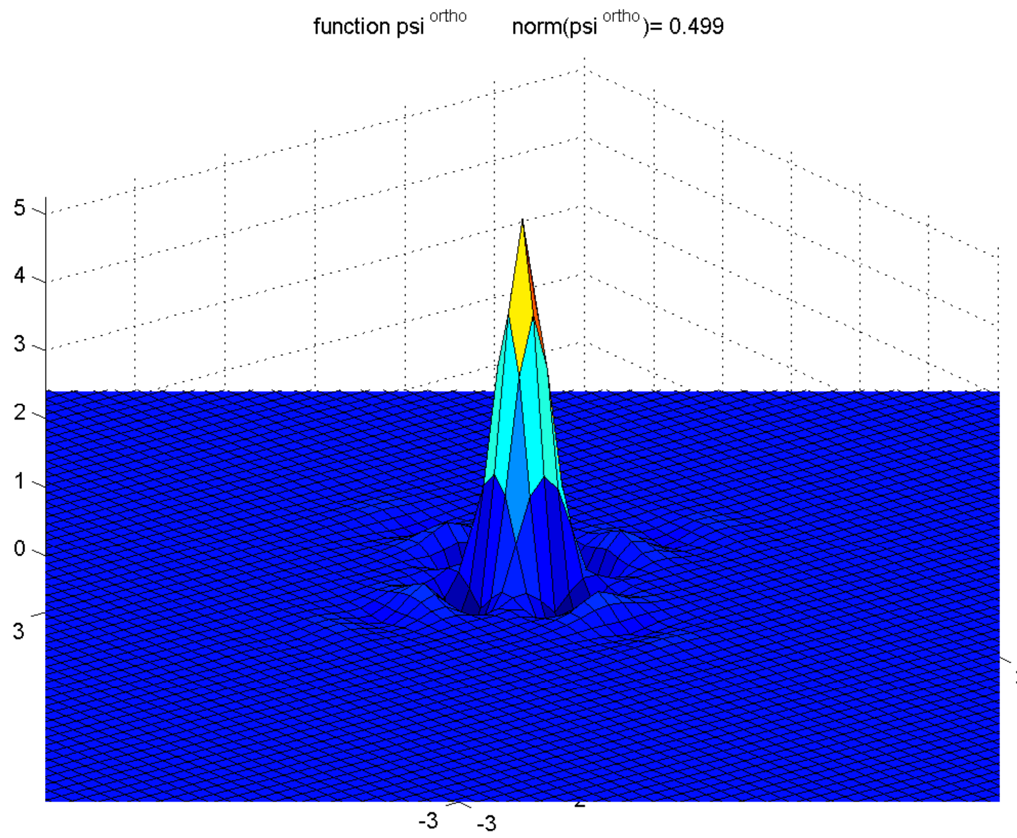
Biharmonic (“thin plate”) L and ψ-spline





Biharmonic (“thin plate”) ψ and $\psi^{\perp}$-spline





(d-dimensional) scaled Whittle-Matérn-Sobolev kernel

Idea

Regularize the operator $-\Delta^m$ of polyharmonic splines
via a regularization of the differential operator:
instead of $(-\Delta)^m$, use the operator $E = (-\Delta + \kappa^2 I)^m$,
κ being some real number("scaled" is for $\kappa \neq 1$).

Its Fourier transform is clearly $\widehat{E}(\omega) = (\|\omega\|^2 + \kappa^2)^m$.

Associated kernel (u function)

$$u_{m,\kappa}(x) = \frac{2^{1-m}}{(m-1)!} \kappa^{d/2-m} \|x\|^{m-d/2} K_{m-d/2}(\kappa \|x\|)$$

where $K_{m-d/2}$ is the order $m - d/2$ Bessel function of the third kind

(note the "mythic" $m - d/2$ constraint,
actually here for ensuring u_m being a continuous function
and the differential operator having a meaning).

Same type of functions generated.

A generalisation of Whittle-Matérn kernel (1/2)

Mira Bozzini, Milvia Rossini, Robert Schaback, 2012

Idea:

Mix different values of κ for each scaled iterated Laplacian operator:
use $E = \prod_{j=1}^{m}(-\Delta + \kappa_j^2 I)$ instead of $(-\Delta + \kappa^2 I)^m$,

We get:

$$\widehat{E}_\kappa(\omega) = \prod_{j=1}^{m}(\|\omega\|^2 + \kappa_j^2)$$

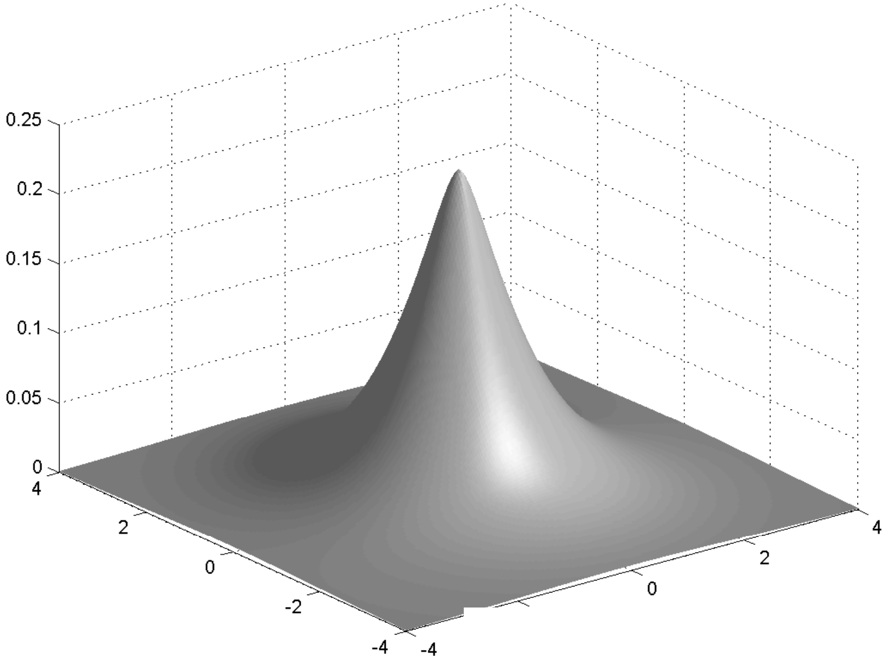
The fundamental solution meets $\widehat{u}_\kappa(\omega) = \prod_{j=1}^{m}(\|\omega\|^2 + \kappa_j^2)^{-1}$
and u_κ is a continuous radial kernel, and is a convolution product of the Kernels u_{1,κ_j} whose Fourier transform is $(\|\omega\|^2 + \kappa_j)^{-1}$ (no computable by this way !).
However we have the following explicit value

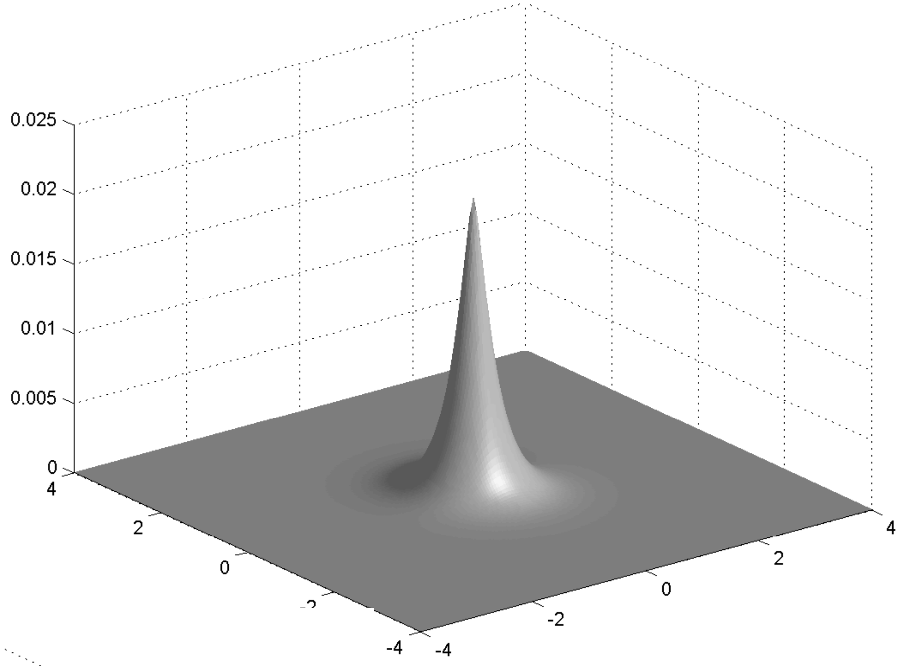
$$u_\kappa(x) = 2^{1-m}(-1)^{m-1}[\kappa_1^2/2, \dots, \kappa_m^2/2]_z \left(\frac{\|x\|}{\sqrt{2z}}\right)^{1-d/2} K_{1-d/2}(\|x\|\sqrt{2z}).$$

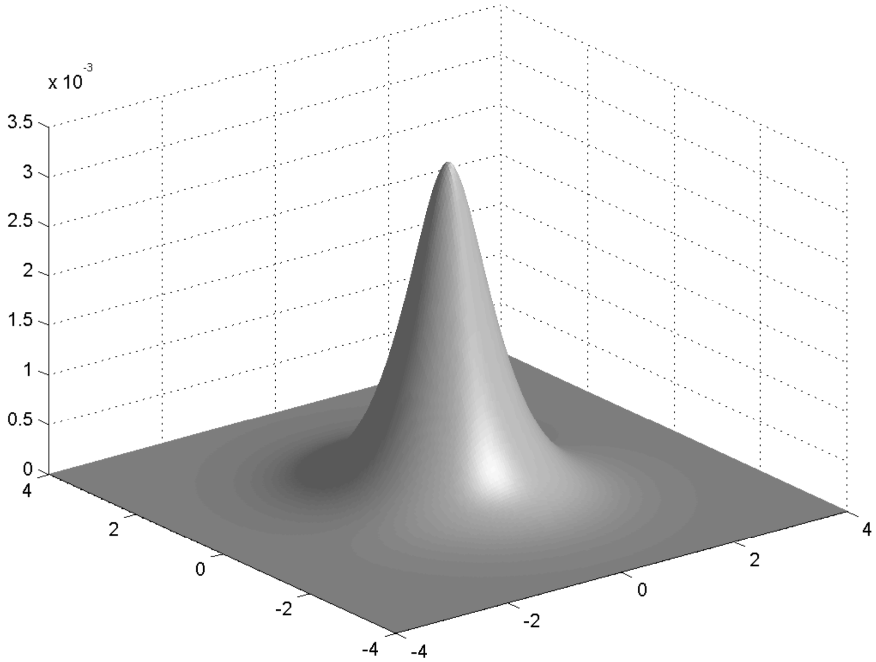
To prove it we use the m-th divided difference relation :

$$(-1)^{m-1}\prod_{j=1}^{m}(s + t_j)^{-1} = [t_1, \dots, t_m]_z(s+z)^{-1}$$

Form of some u-functions







$\kappa = (1\,2),\ (3\,7),\ (2\,3\,4)$

A generalisation of Whittle-Matérn kernel (2/2)

Associated "B-spline"

$$\widehat{\varphi_\kappa}(\omega) \;=\; \prod_{j=1}^{m} \frac{\|\sin\omega\|^2 + \kappa_j^2}{\|\omega\|^2 + \kappa_j^2}$$

"Lagrangian function"

$$\widehat{L_\kappa}(\omega) \;=\; \frac{\prod_{j=1}^{m} (\|\omega\|^2 + \kappa_j^2)^{-1}}{\sum_{\ell\in\mathbb{Z}^d} \prod_{j=1}^{m} (\|\omega - 2\pi\ell\|^2 + \kappa_j^2)^{-1}}$$

Wavelets: $\psi_\kappa \;=\; \left(\prod_{j=1}^{m} (-\Delta + \kappa_j\, I)\right) L_{[\kappa\;\kappa]}$

$$\widehat{\psi_\kappa}(\omega) \;=\; \left(\prod_{j=1}^{m} (\|\omega\|^2 + \kappa_j^2)\right) \widehat{L_{[\kappa\;\kappa]}}(\omega) \;=\; \frac{\left(\prod_{j=1}^{m} (\|\omega\|^2 + \kappa_j^2)\right)}{\sum_{\ell\in\mathbb{Z}^d} \left(\prod_{j=1}^{m} (\|\omega - 2\pi\ell\|^2 + \kappa_j^2\right)}$$

$$\psi_\kappa^\perp(\omega) \;=\; \frac{\left(\prod_{j=1}^{m} (\|\omega\|^2 + \kappa_j^2)\right)}{\sqrt{\sum_{\ell\in\mathbb{Z}^d} \left(\left(\prod_{j=1}^{m} (\|\omega - 2\pi\ell\|^2 + \kappa_j)\right)\right.}}$$

Proof of the expression of the pre-wavelet

Theorem

Let $E = \prod_{j=1}^{m}(-\Delta + \kappa_j^2 I)$.
Let L_2 *be the Lagrangian* E^2*-function, and let* $\psi = E(L_2)$ *(so* $\widehat{\psi}(\omega) = \widehat{E}(\omega)\,\widehat{L_2}(\omega)$*).*
Let $\psi_e = \psi(2\bullet + e) = (E(L^2))(2\bullet + e)$ *where* $e \in \{0,1\}^d \backslash 0^d$.

Then ψ_e *is orthogonal to any cardinal E-function.*
So ψ_e *is a pre-wavelet (semi-orthogonal wavelet).*

Proof

Let $\sigma = \sum_{j\in\mathbb{Z}^d} \lambda_j\, u(\bullet - j)$ be a cardinal E-function (u is such that $E(u) = Dirac$).

Then :

$$
\begin{aligned}
(\sigma\,,\psi_e) &= (\sigma\,, E(L_2(2\bullet + e)\,)\\
&= (E(\sigma)\,, L_2(2\bullet + e)\,)\\
&= (\textstyle\sum_{j\in\mathbb{Z}^d} \lambda_j\, E(u(\bullet - j))\,, L_2(2\bullet + e)\,)\\
&= (\textstyle\sum_{j\in\mathbb{Z}^d} \lambda_j\, Dirac_j\,, L_2(2\bullet + e)\,)\\
&= \textstyle\sum_{j\in\mathbb{Z}^d} \lambda_j\, L_2(2j + e)\\
&= 0.
\end{aligned}
$$

Actually an even more general property (quite general E*, and also for scattered data Lagrangean function* L_2 *and pre-wavelet* $\psi = E(L_2)$*).*

Part 2:

A proposal for a global extension of polyharmonic splines:

functions generated by the differential operator

$$E = \prod_{i=1}^{m} (-\Delta + \kappa_j^2 I)^{\alpha_j}$$

The proposed generalization of polyharmonic splines

Idea:

We want both “tension” and “continuous choice of the order”.
So we choose polyharmonic splines

- regularized by the (extended) Whittle-Matérn coefficients (for tension),
- generalized by real exponents, (for continuous choice of the order and of the shape).

So :

$$E_{\kappa,\alpha} = \prod_{i=1}^{m} (-\Delta + \kappa_j^2 I)^{\alpha_j}$$

Since the differential operator is a mix of the $(-\Delta + \kappa_j^2 I)^{\alpha_j}$, the obtained functions will be a kind of mix of the fractional regularized polyharmonic splines.

We get (with the condition $\sum_{j=1}^{m} \alpha_j > d/2$)

$$\widehat{E}_{\kappa,\alpha}(\omega) = \prod_{i=1}^{m} (\|\omega\|^2 + \kappa_j^2)^{\alpha_j}$$

The fundamental u solution is a radial function which meets

$$\widehat{u}_{\kappa,\alpha}(\omega) = \prod_{j=1}^{m} (\|\omega\|^2 + \kappa_j^2)^{-\alpha_j}$$

Associated functions

Associated "B-function"

$$\widehat{\varphi_{\kappa,\alpha}}(\omega) = \frac{\widehat{E_{\kappa,\alpha}}(\sin\omega)}{\widehat{E_{\kappa,\alpha}}(\omega)} = \frac{\prod_{j=1}^{m}(\|\sin\omega\|^2+\kappa_j^2)^{\alpha_j}}{\prod_{i=1}^{m}(\|\omega\|^2+\kappa_j^2)^{\alpha_j}}$$

"Lagrangian function"

$$\widehat{L_{\kappa,\alpha}}(\omega) = \frac{\prod_{j=1}^{m}(\|\omega\|^2+\kappa_j^2)^{-\alpha_j}}{\sum_{\ell\in\mathbb{Z}^d}\prod_{j=1}^{m}(\|\omega-2\pi\ell\|^2+\kappa_j^2)^{-\alpha_j}}$$

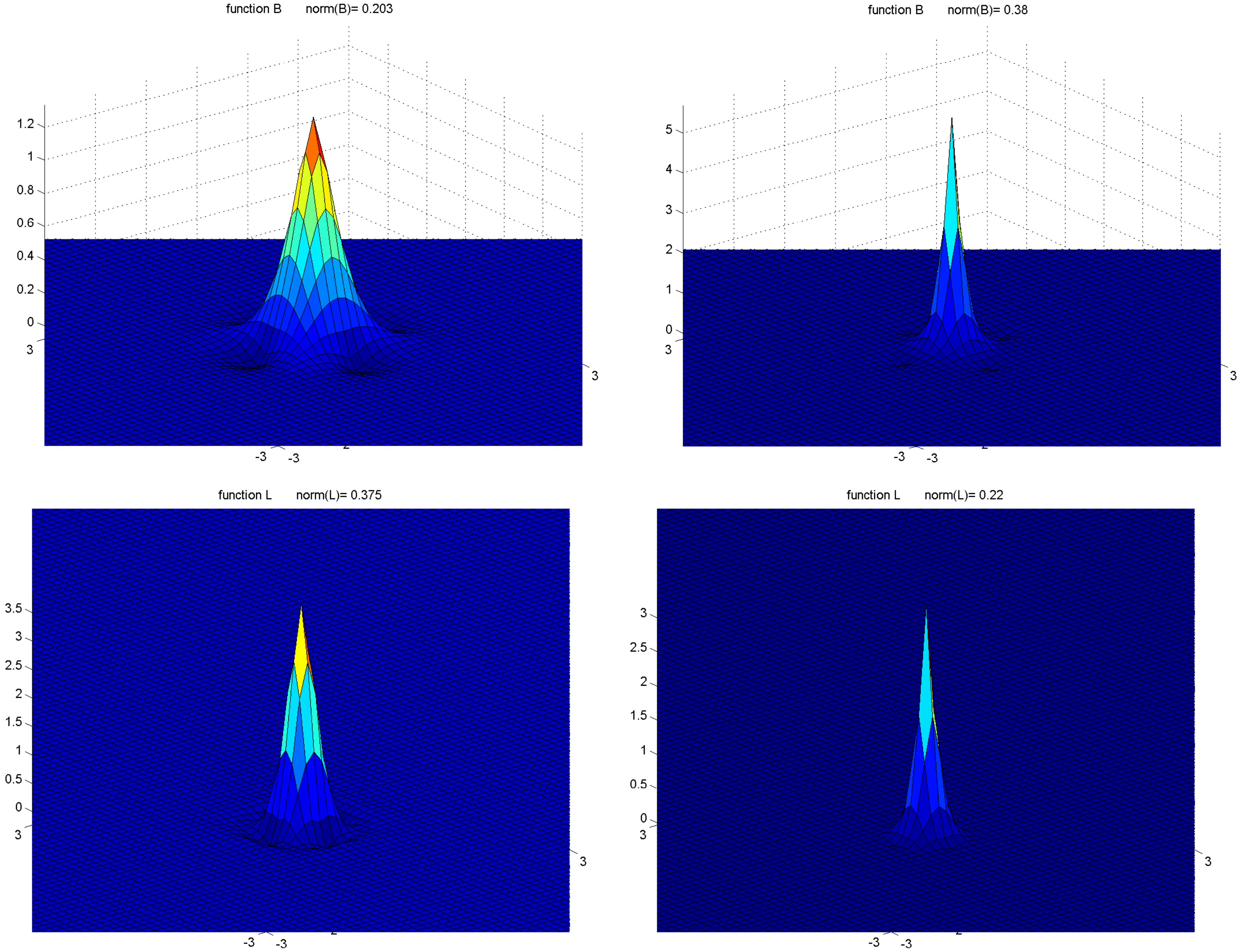
Wavelets: $$\psi_{\kappa,\alpha} = \left(\prod_{j=1}^{m}(-\Delta+\kappa_j^2\, I)^{\alpha_j}\right) L_{[\kappa\ \kappa;\alpha\ \alpha]}$$

$$\widehat{\psi_{\kappa,\alpha}}(\omega) = \left(\prod_{j=1}^{m}(\|\omega\|^2+\kappa_j^2)^{\alpha_j}\right)\widehat{L_{[\kappa\ \kappa\ \alpha\ \alpha]}}(\omega) = \frac{\left(\prod_{j=1}^{m}(\|\omega\|^2+\kappa_j^2)^{\alpha_j}\right)^{-1}}{\sum_{\ell\in\mathbb{Z}^d}\left(\prod_{j=1}^{m}(\|\omega-2\pi\ell\|^2+\kappa_j^2)^{\alpha_j}\right)^{-2}}$$

$$\widehat{\psi^{\perp}_{\kappa,\alpha}}(\omega) = \frac{(\widehat{E}(\omega))^{-1}}{\sqrt{\sum_{\ell\in\mathbb{Z}^d}|\widehat{E}(\omega-2\pi\ell)|^{-2}}} = \frac{\left(\prod_{j=1}^{m}(\|\omega\|^2+\kappa_j^2)^{\alpha_j}\right)^{-1}}{\sqrt{\sum_{\ell\in\mathbb{Z}^d}\left(\prod_{j=1}^{m}(\|\omega-2\pi\ell\|^2+\kappa_j)^{\alpha_j}\right)^{-2}}}$$

Examples of B- and L- functions

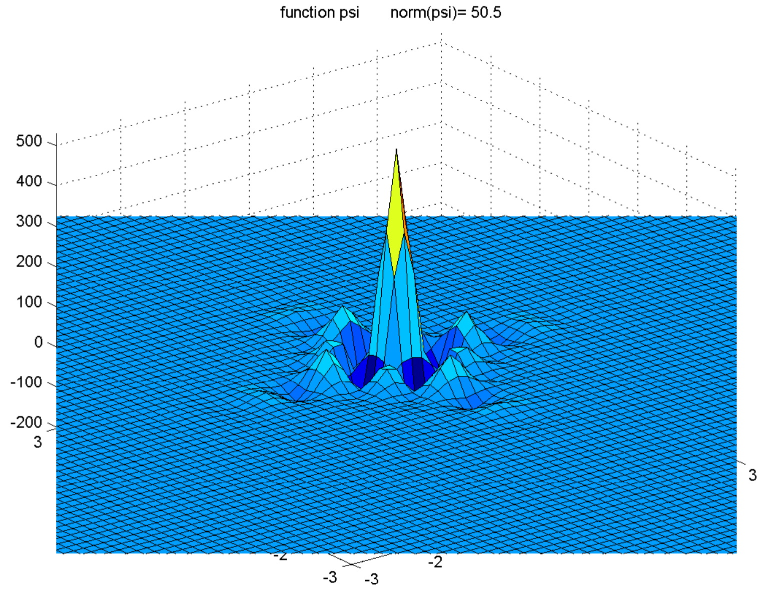
B-function

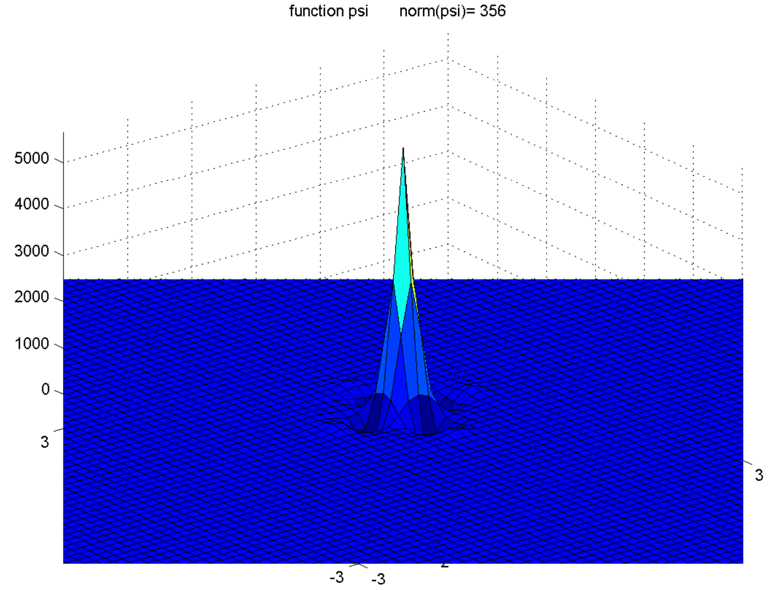


L-function

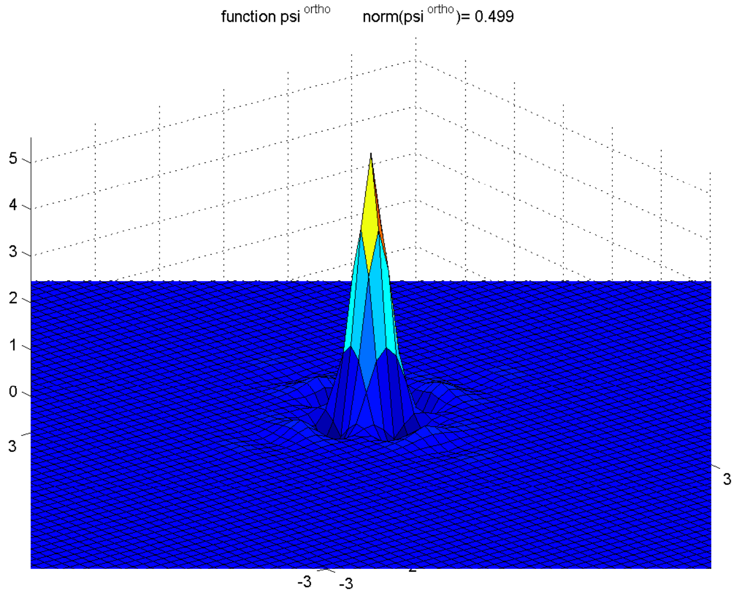
$\kappa = (1\ 2)$ $\kappa = (3\ 7)$

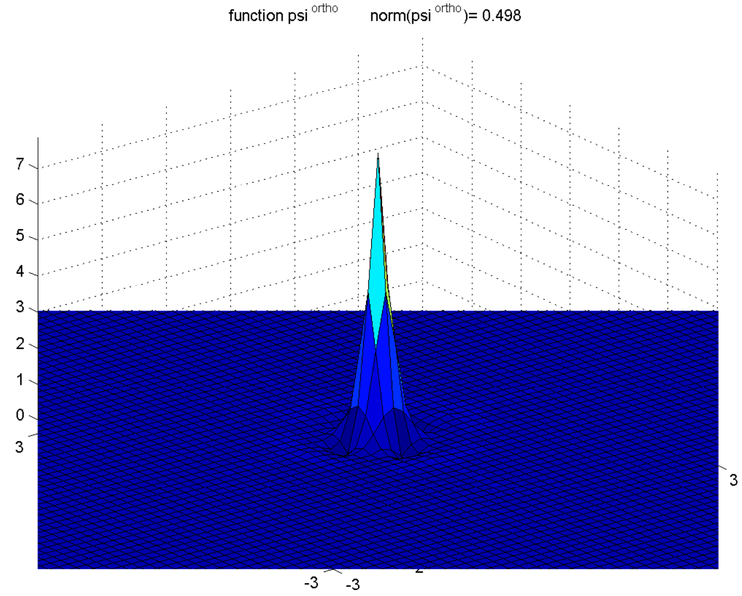
Examples of ψ- and $\psi^\perp$ functions





ψ-function





$\psi^\perp$ function

$\kappa = (1\ 2)$ $\kappa = (3\ 7)$

Minimization property, when all α_j are integer

Let γ be such that $\sum_{i=0}^{m} \gamma_i\, x^i = \prod_{j=1}^{m} (x+\kappa_j^2)$.

γ_i is positive and is the sum of all product of $m-i$ different κ_j^2.

In particular : $\gamma_0 = \prod_{j=1}^{m} \kappa_j^2, \quad \gamma_1 = \sum_{i=1}^{m} \prod_{j\neq i} \kappa_j^2, \quad \gamma_{m-1} = \sum_{j=1}^{m} \kappa_j^2, \quad \gamma_m = 1\ ;\ \gamma_j > 0.$

Let $\mathcal{H}^\kappa$: $(f\,,g)_\kappa = \int_{\mathbb{R}^d} \sum_{j=0}^{m} \gamma_j\,(D^j f \cdot D^j g)$ and $|f|_\kappa = \left(\int_{\mathbb{R}^d} \sum_{j=0}^{m} \gamma_j\, \|D^j f\|^2\right)^{\frac{1}{2}}$

Theorem
Let A be a set of elements in $\mathbb{R}^d$, and $(y_a)_{a\in A}$ some associated real numbers. If A is an infinite set, suppose furthermore that there exists $f \in \mathcal{H}^\kappa$ such that $\forall a \in A\ ,\ \ f(a) = y_a$.

Then the set of all functions f in $\mathcal{H}^\kappa$ meeting $\forall a\in A\ ,\ \ f(a) = y_a$ has a unique element, denoted $\sigma_{A,\kappa}$ with minimal (semi-)norm $|f|_\kappa$.

Besides, there exist real numbers $(\lambda_a)_{a\in A}$ such that $\sigma_{A,y}$ meets the relation

$$E(\sigma_{A,y}) = \sum_{a\in A} \lambda_a \delta_a, \qquad \text{so} \qquad \sigma_{A,y} = \sum_{a\in A} \lambda_a u_\kappa(x-a) + p_{\ell-1}\ ,$$

where ℓ is the number of κ_j being 0 and $p_{\ell-1}$ is some polynomial in $\mathbb{P}_{\ell-1}$.

Role and influence of κ : tension coefficients

Theorem (direct consequence of previous theorem)

Let σ_A be in the form $\sigma_A = \sum_{a\in A} \lambda_a u_\kappa(\bullet - a) + p_{\ell-1}$ (1),
and let $y = (y_a)_{a\in A}$ such that $y_a = \sigma_A(a)$.

Then σ_A minimizes $|f|_\kappa$ on all the functions $f \in \mathcal{H}^\kappa$ such that $\forall a \in A$, $f(a) = y_a$.

Discussion

Since $|f|_\kappa = \left(\int_{\mathbb{R}^d} \sum_{j=0}^{m} \gamma_j \, \|D^j f\|^2\right)^{\frac{1}{2}}$, we see that any function in the form (1) minimizes a quantity which is a linear combination of various polyharmonic semi-norms, the weights of the linear combination being the γ_j, which are connected to the κ_j by the relation $\sum_{i=0}^{m} \gamma_i \, x^i = \prod_{j=1}^{m} (x + \kappa_j^2)$.

Since it is known that a polyharmonic spline is all the more oscillating than its order is high, we can interpret γ_j as tension coefficients, giving all the more tension than j is small. Intuition of the impact of a vector κ is quite easy when all but one κ_j are zero, but is not so easy otherwise.

Role and influence of real valued α_j's

Very easy when only one α_j is non integer and when the associated κ_j is zero : we have real order polyharmonic splines under tension, with tension parameters as explained above for integer α_j's.

As said before, we know that real order polyharmonic splines are somewhere between the polyharmonic spline of the integer part of the order and of the integer part plus 1.

Otherwise, we need work directly on the Fourier transform of E, and the minimized quantity is less intuitive.

There is still work to do to improve intuition about that.

Two remarks

On B-functions

Note that for (one dimensional) polynomial splines, the B-spline is non-negative.

This is NOT true for many other "B-functions", as some tension B-splines, polyharmonic B-splines and all their d-dimensional extensions.

We claim that the important property is NOT non-negativity, but "positive definite functions, as they all are since $\widehat{\varphi}$ is non-negative (in all extensions we showed here).

On the strategy

Starting on the differential operator and/or its Fourier transform is an interesting way to work on, and gives other possibilities, depending on what we want to obtain.

An operator involving iterated Laplacian operator will give radial basis functions. Of course appropriate properties for having a MRA have to be checked.

For example if we want "something between quintic and linear" we can use $\kappa = (0\ \rho_1\ \rho_2)$ (which gives $E = D^6 + (\rho_1^2 + \rho_2^2)D^4 + \rho_1^2\rho_2^2 D^2$ in one dimension) and so get $\widehat{E}(\omega) = -\omega^6 + (\rho_1^2 + \rho_2^2)\omega^4 - \rho_1^2\rho_2^2\omega^2$ (or equivalent with Δ and $\|\omega^2\|$ in many dimensions), but we also can try to minimize $\int_{\mathbb{R}} (f^{(3)}(x))^2 + \rho^2 (f'(x))^2 dx$, which gives $E = D^6 + \rho^2 D^2$ and $\widehat{E}(\omega) = -\omega^6 - \rho^2\omega^2$ (or equivalent with Δ and $\|\omega\|^2$ in many dimensions), which is not included in the presented extension (not done !).

Computational considerations

Extensive use of the fast Fourier transform

All the functions are given by their Fourier transform.

The whole computation is done via FFT and IFFT.

For the basis functions (φ, ψ, $\psi^\perp$, L: sampling the Fourier transform of the function, and then IFFT of the so-obtained (d-dimensional) vector.

the longest computation is for computing the sample of the Fourier transform of the functions.

For spline functions in the form $\sum_{i \in \mathbb{Z}^d} y_j \, B(\bullet - j\,h)$, we compute $\widehat{y}(\omega)$ (via the FFT), and then we compute $IFFT(\widehat{y}\,\widehat{B})$. Note that this is no more complicated for any type of extension presented here.

As usual, there are some side effects for boundly supported vectors or functions.

Extension to scattered data analysis

Surprisingly this is possible, still by using this type of kernel.
However everything is not yet done.

The main difficulty is to choose the appropriate points close to the point where we want to discretize the derivatives. But we can define the basis functions (one for each centre !) via the Fourier transform, and go on pretty well. However this need still work theoretical as well as numerical... Besides there are some true normalization problems

A classical test : Lena

We use the associated filters on one Lena's eye :

and obtain :

$\kappa = (1\ 2)$

$\kappa = (3\ 7)$

Enjoy your reseach,

Enjoy your life,

...and take care !

Some curves for $\kappa = [0; .5; 1; 2]$