Divided differences

Tomas Sauer

Lehrstuhl für Mathematik, Schwerpunkt Digitale Bildverarbeitung
FORWISS
Universität Passau

Fakultät für Informatik und Mathematik

MAIA 2013, Erice, September 26, 2013

In part joint work with J. Carnicer (Zaragoza)

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Un' occassione di raccoliere in Sicilia ...

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Un' occassione **DI RAC***coliere in Sicilia ...*

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Passau

Where three rivers meet . . .

Passau

…lies the "Bavarian Venice" …

Passau

. . . lies the “Bavarian Venice” . . .

Passau

…lies the "Bavarian Venice" …

Passau

. . . with an “Underwater University” . . .

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Passau

. . . and great students

Passau

...and great students

Interpolation

Interpolation/recovery problem

Given **sites** Ξ and **data** $y \in \mathbb{R}^{\Xi}$ find f such that

Well known …

1. Many, many solutions.
2. Try **linear function space** of dimension $\#\Xi$.
3. $\Xi \subset \mathbb{R}$: polynomials of degree at most $\#\Xi - 1$ are perfect.
4. 1D: Only a matter of **counting** – no geometry.

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Different Points of View

Point set constructions

Given a space $\mathscr{P} \subset \Pi$ find sites Ξ such that there always exists a unique $p \in \mathscr{F}$ with $p(\Xi) = f(\Xi)$.

Point set constructions

Find subspace $\mathscr{P} \subset \Pi$ such that for any Ξ with $\#\Xi = N$ there exists $p \in \mathscr{P}$ with $p(\Xi) = f(\Xi)$.

Space constructions

Given Ξ find $\mathscr{P}$ such that there always exists $p \in \mathscr{P}$ with $p(\Xi) = f(\Xi)$.

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Given a space $\mathscr{P} \subset \Pi$ find sites Ξ such that there always exists a unique $p \in \mathscr{F}$ with $p(\Xi) = f(\Xi)$.
Answers: Chung–Yao, GPL, ...

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Answers: Π_{N-1}

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Find smallest subspace $\mathscr{P} \subset \Pi$ such that for any Ξ with $\#\Xi = N$ there exists $p \in \mathscr{P}$ with $p(\Xi) = f(\Xi)$.
Answers: Π_{N-1}, Π_{2n-1}, $\binom{n+2}{2} = N$, for $s = 2$.

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Answers: Buchberger–Möller, deBoor–Ron, ideal remainders

Notation

Polynomials

1. Π polynomials = **finite** sums
$$p(x) = \sum_{|\alpha| \le n} p_\alpha \, x^\alpha.$$
2. $\deg p := n$.
3. **Monomials** or **terms** of degree $k, \dots, n$: **row** vectors
$$\mathbf{x}^{k:n} := \left((\cdot)^\alpha \ : \ k \le |\alpha| \le n \right), \qquad \mathbf{x}^n = \mathbf{x}^{n:n}$$
4. Coefficients as **column** vectors $\mathbf{p}_k = (p_\alpha \ : \ |\alpha| = k)$.
5. Polynomials: $p(x) = \sum_{k=0}^{n} \mathbf{x}^k \mathbf{p}_k$.

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Correctness

A subspace $\mathscr{P} \subset \Pi$ is called **correct** for $\Xi \subset \mathbb{R}^s$ if for any $f : \Xi \to \mathbb{R}$ there exists **unique** $L_\Xi f \in \mathscr{P}$ such that $L_\Xi f(\Xi) = f(\Xi)$.

Correctness for Π_n

Vandermonde matrix

$$x^{0:n}(\Xi) = \left(\xi^\alpha : \begin{array}{c} \xi \in \Xi \\ |\alpha| \le n \end{array} \right)$$

Then:

$$L_\Xi f = x^{0:n} \, p,$$

Fundamental "Theorem"

Correctness = nonsingularity of Vandermonde matrix.

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Interpolation Spaces

Degree of a subspace

$$\deg \mathscr{P} = \max\{\deg p \;:\; p \in \mathscr{P}\}.$$

Degree reducing interpolation

$$f \in \Pi \qquad \Rightarrow \qquad \deg L_\Xi f \le \deg f.$$

Minimal degree interpolation space

$$\mathscr{Q} \text{ interpolation space} \qquad \Rightarrow \qquad \deg \mathscr{Q} \ge \deg \mathscr{P}.$$

Facts

1. Degree reducing is always minimal degree.
2. None of them is unique for general Ξ.
3. Different elimination strategies for $x^{0:n}(\Xi)$.

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The Two Faces of Polynomials

The **linear** part

1. Computation of interpolant: solve *Vandermonde system*.

But . . .

1. Polynomials can be *multiplied*.
2. Polynomial algebra . . .

T. Pratchett, *Jingo*

There's al–gebra. That's like sums with letters. For . . . for people whose brains aren't clever enough for numbers, see?

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Ideals

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1. **Ideal** $\mathscr{I}$:
2. $\mathscr{I}(\Xi) := \{f \in \Pi \, : \, f(\Xi) = 0\}$.
3. **Ideal generated by** $F \subset \Pi$:
4. $\mathscr{F}$ **basis** of $\mathscr{I} = \langle F \rangle$.

Hilbert's Basissatz

Any polynomial ideal has a finite basis.

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Hilbert's Basissatz

Any polynomial ideal has a finite basis.

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1. **Ideal** $\mathscr{I}$: $\mathscr{I} + \mathscr{I} = \mathscr{I}$ and $\mathscr{I} \cdot \Pi = \mathscr{I}$.
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From Interpolation to Bases

Construction of basis

1. $\mathscr{P}$ **degree reducing interpolation space** for Ξ.
2. P basis for $\mathscr{P}$.
3. Set $P^* = \{(\cdot)_j p \; : \; p \in P, j = 1, \ldots, s\}$.
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Even better

Basis is **H–basis**:

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$$\mathscr{I}(\Xi) \ni g = \sum_{f \in F_\Xi} g_f f, \qquad \deg g_f f \le \deg g.$$

Implicit Interpolation

The case $s = 1$

1. $\omega = \prod_{\xi \in \Xi} (\cdot - \xi)$.
2. Division with remainder:
3. $L_\Xi f := r$ interpolation polynomial.

Algebraic interpretation

1. Principal ideal: $\mathscr{I}(\Xi) = \langle \omega \rangle$.
2. Ideal + Quotient Space.

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Implicit Interpolation II

Division with remainder

1. Division by *set* F:

$$g = \sum_{f \in F} g_f f + \nu_F(g).$$

2. Ideal + normal form.
3. Normal form is
   1. interpolant.
   2. unique if F is H–basis.

Constructive chain

Remark

All degree reducing interpolants can be constructed this way.

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Space Decompositions

Theorem

$\mathscr{P}$ is degree reducing interpolation space if and only if

$$\Pi_k = (\mathscr{P} \cap \Pi_k) \oplus (\mathscr{I}(\Xi) \cap \Pi_k), \qquad k = 0, \dots, n := \deg \mathscr{P}.$$

Theorem

$\mathscr{P}$ is degree reducing interpolation space if and only if

1. $\Xi = \Xi_0 \cup \cdots \cup \Xi_n$.
2. There exists **Newton basis** $n^k \in \Pi^{\#\Xi_k}$, $k = 0, \dots, n$, such that

Space Decompositions

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Two bases

Assumption (for simplicity)

$$\mathscr{P} = \Pi_n.$$

Monic basis

$$m^k = \left((\cdot)^\alpha - L_{\Xi_{0:k-1}} (\cdot)^\alpha \ : \ |\alpha| = k \right\}$$

Newton basis

$$n^k = m^k \, m_k \left(\Xi_k \right)^{-1}.$$

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Divided Differences

Definition

$[\Xi]f = [\Xi_{0:n}]f =$ leading monomial coefficients in $L_\Xi f$:

Properties

1. Structure like derivative
2. Depends only on $f(\Xi_{0:k})$.
3. Symmetric in $\Xi_{0:k}$ and affine invariant.
4. Annihilates Π_{k-1}.
5. Duality: $I = [\Xi_{0:k}]\, x^k$

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$[\Xi_{k:n}]f$ = monomial coefficients in $L_\Xi f$:

$$L_\Xi f = \sum_{k=0}^{n} \boldsymbol{m}^k \, [\Xi_{0:k}] f.$$

Natural idea, e.g. [Rabut2000].

Properties

1. Structure like derivative
2. Depends only on $f(\Xi_{0:k})$.
3. Symmetric in $\Xi_{0:k}$ and affine invariant.
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Derivatives and Splines

Theorem

de Boor's "divided" difference

$$[\Xi]_S f := [\Xi; \Xi D]_S f := \int_\Xi D_{\xi_1 - \xi_0} \cdots D_{\xi_N - \xi_{N-1}} f.$$

1. Simplex spline integral.
2. Appears in *Kergin interpolation*.
3. Building block for spline representation of divided difference.

Derivatives and Splines

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$$[\xi + h\,\Xi_{0:k}]f$$

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Notation

1. **Path:** $\Theta \in \Xi_0 \times \cdots \times \Xi_n$
2. **Paths to** $\xi \in \Xi_n$**:** $P(\xi) = \{\Theta \ : \ \theta_n = \xi\}$.
3. **Newton value** of Θ:

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$$[\Xi_{0:n}] f = \boldsymbol{m}^n (\Xi_n)^{-1} \left(\sum_{\Theta \in P(\xi)} n(\Theta) \ [\Theta]_S f \ : \ \xi \in \Xi_n \right).$$

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Recurrence Relation

Lemma

For $\xi \in \Xi_k$ there exists $\xi' \in \Xi_{k-1}$ such that

$$\Xi_{0:k-1}^{\xi} := \Xi_{0:k-1} \setminus \{\xi'\} \cup \{\xi\}$$

is correct for Π_{k-1}.

Theorem

The divided difference satisfies

$$[\Xi_{0:n}] f = m^n (\Xi_n)^{-1} \left(\sum_{\xi \in \Xi_n} M_\xi \left[\Xi_{0:n-1}^{\xi} \right] f - M [\Xi_{0:n-1}] f \right)$$

where

$$M_\xi = e_\xi m^{n-1}(\xi), \qquad M = m^{n-1}(\Xi_n).$$

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Leibniz Formula

Background

1. Joint work with J. Carnicer.
2. Formulas and understanding of $[\Xi_{0:n}]\,(fg)$.

Observation and definition

1. Simple computation:

$$[\Xi_{0:n}]\,(fg) = \sum_{k=0}^{n} [\Xi_{k:n}]' g \, [\Xi_{0:k}] f,$$

2. **Complementary divided difference $[\Xi_{k:n}]' g$.**
3. $\#\Xi_{0:n} \times \#\Xi_{0:k}$–matrix valued.
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Complementary Divided Difference

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Interpretation

Complementary divided difference $[\Xi_{j:k}]'$

1. describes interpolation at $\Xi_{k:n} = \Xi_{0:n} \setminus \Xi_{0:k-1}$
2. $[\Xi_{k:n}]'f$ depends on $f(\Xi_{k:n})$.

The case $s = 1$

$$[\Xi_{0:k}]'g$$

Definition

Divided Difference $[\Xi_{k:n}]f := [\Xi_{0:n}](m^k f)$.

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$$[\Xi_{0:k}]'g = [\xi_0, \dots, \xi_n] \left(g \prod_{j=0}^{k-1} (\cdot - \xi_j) \right) = [\xi_k, \dots, \xi_n]\, g = [\Xi_{k:n}]g$$

Definition

Divided Difference $[\Xi_{k:n}]f := [\Xi_{0:n}] \left(\boldsymbol{m}^k f \right)$.

Divided Difference

Interpretation

Complementary divided difference $[\Xi_{j:k}]'$

1. describes interpolation at $\Xi_{k:n} = \Xi_{0:n} \setminus \Xi_{0:k-1}$ by means of $\langle \boldsymbol{m}^k \rangle$.
2. $[\Xi_{k:n}]'f$ depends on $f\left(\Xi_{k:n}\right)$.

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Divided Difference

Leibniz rule

$$[\Xi_{k:n}]\,(fg) = \sum_{j=k}^{n} [\Xi_{j:n}]g\,[\Xi_{k:j}]f.$$

Covers …

1. …univariate case.
2. …tensor product case:

$$[\Xi_{\alpha:\beta}](fg) = \sum_{\gamma=\alpha}^{\beta} [\Xi_{\alpha:\gamma}]f\,[\Xi_{\gamma:\beta}]g,$$

where

$$[\Xi_{\alpha:\beta}]f = [\xi_{\alpha_1,1},\dots,\xi_{\beta_1,1};\dots;\xi_{\alpha_s,s},\dots,\xi_{\beta_s,s}]f.$$

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Another product rule

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Theorem

$$[\Xi](fg) = \sum_{j+k\ge n} \left([\Xi_{0:j}]f\right)^T M_{jk}\, [\Xi_{0:k}]g,$$

with the *tensor*

$$M_{jk} = \left([\Xi_{k:n}](\boldsymbol{m}^j)^T\right) \in \mathbb{R}^{\#\Xi_j \times \#\Xi_k \times \#\Xi_n}.$$

Corollary

$$[\Xi]_\gamma(fg) = \sum_{\alpha+\beta=\gamma} [\Xi_{0:|\alpha|}]_\alpha f\, [\Xi_{0:|\beta|}]_\beta g + R_\gamma(\Xi, f, g)$$

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Leibniz for partial derivatives.

Interpolation in Ideals

Goal

1. Interpretation of "generalized" divided difference $[\Xi_{k:n}]f$.
2. Consider L_Ξ as operator on Π.

The ideal

$$\mathscr{I} = \mathscr{I}(\Xi_{0:k-1})$$

The interpolant

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The interpolant

$$L'_{\Xi_{k:n}} \; : \; \mathscr{I} \to \mathscr{I},$$

Interpolation in Ideals

UNIVERSITÄT PASSAU

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The ideal

$$\mathscr{I} = \mathscr{I}(\Xi_{0:k-1}) = \left\{ \boldsymbol{m}^k \boldsymbol{h}^T \; : \; \boldsymbol{h} = (h_\alpha : |\alpha| = j) \in \Pi^{\#\Xi_k} \right\}.$$

The interpolant

$$L'_{\Xi_{k:n}} \; : \; \mathscr{I} \to \mathscr{I}, \qquad L'_{\Xi_{k:n}} f = \sum_{j=k}^{n} \boldsymbol{m}^j \operatorname{trace}\left([\Xi_{k:j}] \boldsymbol{h}^T \right).$$

Summary

Conclusions

1. Leading coefficient of L_Ξ is the "right" divided difference.
2. Various properties extend.
3. Complicated formulas simplify for $s = 1$.
4. Extension via complementary difference.

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We earned our coffee!