On smooth spline spaces and quasi-interpolants over Powell-Sabin triangulations

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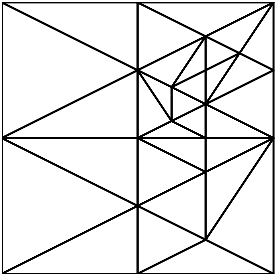
MAIA Conference
Erice, September 25–30, 2013

Outline

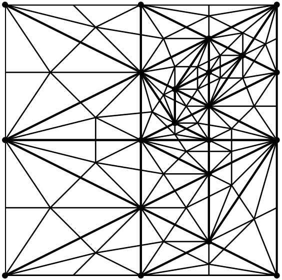
Introduction

Triangulation with Powell-Sabin split [Powell & Sabin, TOMS 1977]

- Every triangle is split into six subtriangles
- E.g., incenter as split point



$\Rightarrow$



Introduction

Univariate B-spline representation

- Basis: local support, convex partition of unity
- Control points (CAGD)
- Easy manipulation:
  stable evaluation [e.g. de Boor], differentiation, integration
- ...

Bivariate B-spline representation

- Smooth splines on Powell-Sabin triangulations
- Basis with similar properties as univariate case
- Construction of quasi-interpolations (using blossoming)

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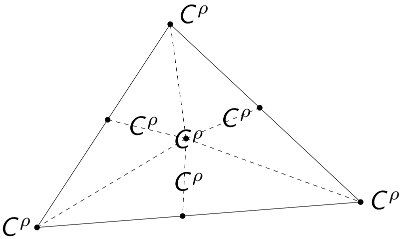
Smooth Powell-Sabin (PSr) splines

PSr-spline space

We consider piecewise polynomials of degree d with global C^r-continuity and C^ρ-supersmoothness at some points and edges, defined on a triangulation Δ with PS-split Δ^*

$$\mathbb{S}_d^{r;\rho}(\Delta^*) = \{s \in C^r(\Omega) : s|_{T^*} \in \mathbb{P}_d,\ T^* \in \Delta^*; \\ s \in C^\rho(W),\ W \in (\mathcal{V} \cup \mathcal{Z}^*);\ s \in C^\rho(e),\ e \in \mathcal{E}^*\}$$



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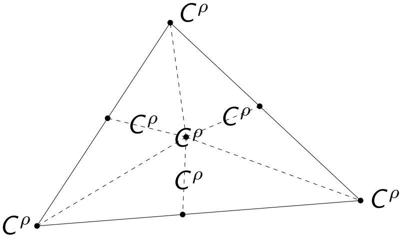

PSr-splines: for a given r,
$d = 3r - 1$, $\rho = 2r - 1$

$r = 1$: [Powell & Sabin, 1977, ...]
$r = 2$: [Sablonnière, 1987, ...]
$r > 2$: [S., 2013]

Smooth Powell-Sabin (PSr) splines

PSr-spline space

- Let n_v vertices and n_t triangles in Δ; let $N_v = \binom{2r+1}{2}$, $N_t = \binom{r}{2}$
- Dimension equals $N_v n_v + N_t n_t$
- Interpolation problem: PSr-spline s is uniquely defined by

$$D_x^a D_y^b s(V_l) = f_{x^a y^b, l}, \qquad l = 1, \dots, n_v, \quad 0 \le a+b \le 2r-1,$$
$$D_x^a D_y^b s(Z_m) = g_{x^a y^b, m}, \qquad m = 1, \dots, n_t, \quad 0 \le a+b \le r-2,$$

  for any given set of $f_{x^a y^b, l}$-values and $g_{x^a y^b, m}$-values.
- Basis?
  - N_t functions $B^t_{k,j}(x, y)$ related to each triangle T_k
  - N_v functions $B^v_{i,j}(x, y)$ related to each vertex V_i

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- Basis?
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- Basis?
  - B-spline-like basis [S., 2010, 2013]
  - local support + nonnegativity + partition of unity

Normalized basis

B-spline related to a triangle T_k

- The B-spline $B^t_{k,j}(x,y)$ is the solution of the interpolation problem

$$g_{x^a y^b,k} = \beta^{ab}_{k,j} \neq 0; \qquad g_{x^a y^b,m} = 0, \quad m \neq k; \qquad f_{x^a y^b,l} = 0$$

(local support)

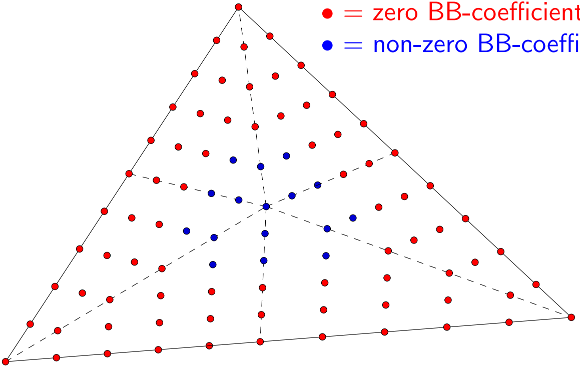
- The values of $\beta^{ab}_{k,j}$ are determined via Bernstein-Bézier representation of B-spline

(nonnegativity)

Normalized basis

B-spline related to a triangle T_k

Example $r = 2, d = 5$:



polynomial of degree $2r - 1$
with single BB-coeff $b_{111} = 1$
and other BB-coeffs $b_{\kappa\mu\nu} = 0$

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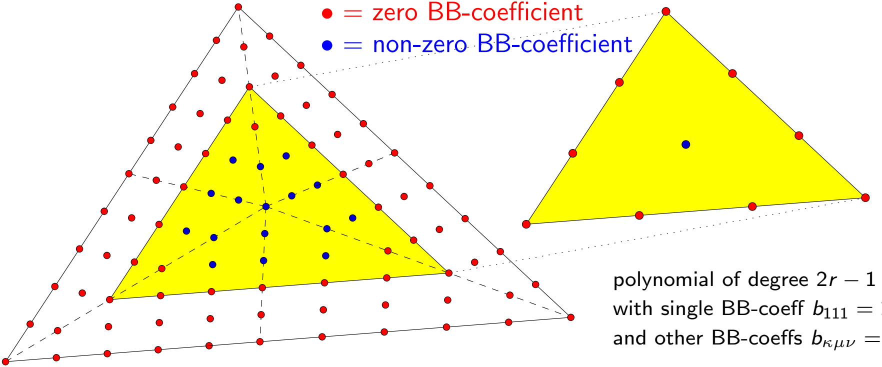

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Normalized basis

B-spline related a vertex V_i

- Let M_i be the molecule of vertex V_i.
  The B-spline $B^v_{i,j}(x,y)$ is the solution of the interpolation problem

$$f_{x^a y^b, i} = \alpha^{ab}_{i,j}; \qquad f_{x^a y^b, l} = 0, \quad l \neq i$$
$$g_{x^a y^b, m} = \beta^{ab}_{i,j}, \quad T_m \in M_i; \qquad g_{x^a y^b, m} = 0, \qquad T_m \notin M_i$$

  (Local support)

- Given $\left\{\alpha^{ab}_{i,j},\ 0 \leq a+b \leq 2r-1\right\}$, the values of $\beta^{ab}_{i,j}$ are determined via Bernstein-Bézier representation of B-spline

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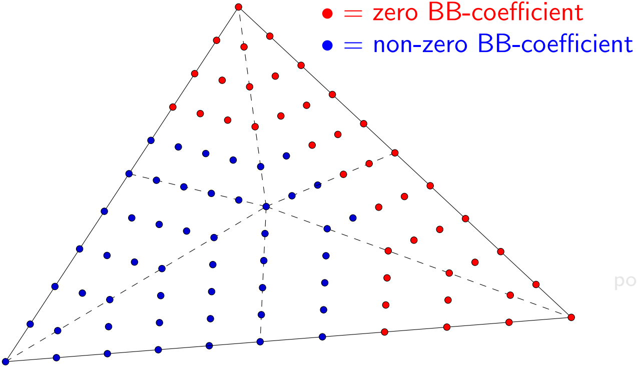

polynomial of degree $2r - 1$

given by $\alpha_{i,j}^{ab}$

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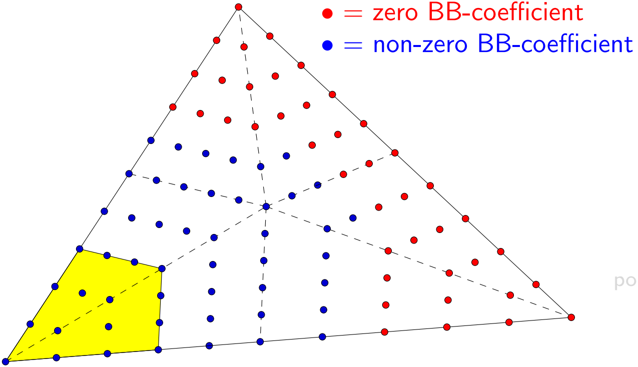

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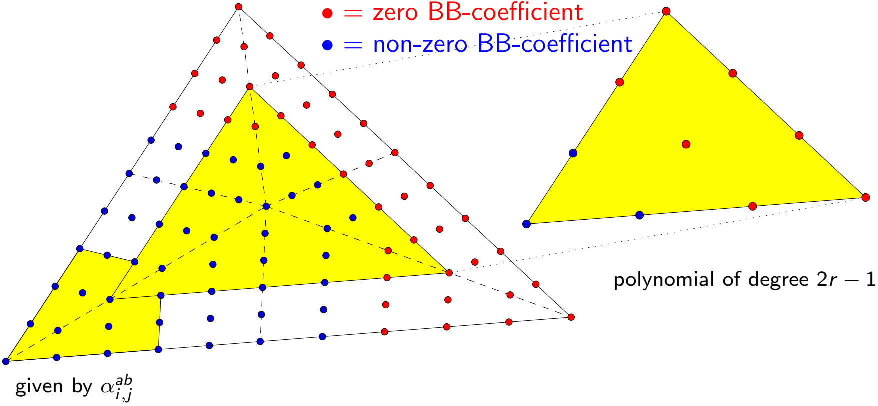


Normalized basis

B-spline related a vertex V_i

- For each V_i, choose a PS-triangle t_i
- Choose

$$\alpha_{i,j}^{ab} = \frac{\binom{3r-1}{a+b}}{\binom{2r-1}{a+b}} (\theta_i)^{a+b} D_x^a D_y^b \mathfrak{B}_{\kappa\mu\nu}^{2r-1}(V_i),$$

with $\mathfrak{B}_{\kappa\mu\nu}^{2r-1}(x,y)$ a Bernstein polynomial defined on t_i, for some $\kappa+\mu+\nu = 2r-1$

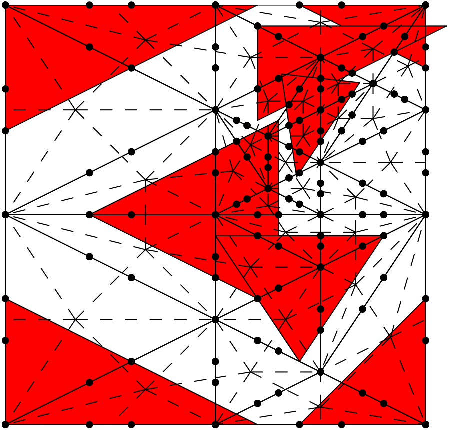
(partition of unity)

Normalized basis

B-spline related a vertex V_i

- Each PS-triangle t_i must contain PS-points:
  - vertex V_i
  - points $(1-\theta_i)V_i + \theta_i V_l$, for any V_l in M_i

  (nonnegativity)
- All Bézier ordinates of B-spline are nonnegative
- Choose small PS-triangles



Normalized basis

B-spline representation

$$s(x,y) = \sum_{i=1}^{n_v} \sum_{j=1}^{N_v} c_{i,j}^v \, B_{i,j}^v(x,y) + \sum_{k=1}^{n_t} \sum_{j=1}^{N_t} c_{k,j}^t \, B_{k,j}^t(x,y)$$

- Stable evaluation through sequence of convex combinations
  $\Rightarrow$ conversion to BB-form + de Casteljau algorithm
- Control points associated to vertices and triangles
  $\Rightarrow$ organized in local Bézier nets
- How to construct efficient quasi-interpolants?
  $\Rightarrow$ blossoming

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Blossoming

Blossom of polynomials

- Given polynomial p_d of degree d
- Characterization of blossom $\mathcal{P}[p_d](P_1, \ldots, P_d)$
  - symmetric: it does not change under permutation of arguments
  - multi-affine: affine in each of its d arguments
  - diagonal property: $p_d(P) = \mathcal{P}[p_d](P, \ldots, P)$
- Compact way to describe subdivision, derivatives, ...
- Notation:

$$\mathcal{P}[p_d]\Bigg(\underbrace{P_1, \ldots,}_{a_1 \ times} \underbrace{P_2, \ldots,}_{a_2 \ times} \underbrace{P_3, \ldots}_{a_3 \ times}\Bigg) = \mathcal{P}[p_d](P_1[a_1], P_2[a_2], P_3[a_3])$$

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Quasi-interpolation

Smooth PSr Quasi-interpolation

$$\mathcal{Q} f(x,y) = \sum_{i=1}^{n_v} \sum_{j=1}^{N_v} c_{i,j}^v \, B_{i,j}^v(x,y) + \sum_{k=1}^{n_t} \sum_{j=1}^{N_t} c_{k,j}^t \, B_{k,j}^t(x,y)$$

- At vertex V_i:
  - PS-triangle t_i with points $Q_{i,1}, Q_{i,2}, Q_{i,3}$
  - set $\widehat{Q}_{i,j} = \frac{\theta_i - 1}{\theta_i} V_i + \frac{1}{\theta_i} Q_{i,j}$
  - choose a (local) polynomial projector $\mathcal{I}_{i,j} f$

    $\Rightarrow$ Taylor polynomial, Lagrange polynomial interpolation, ...
  - set $c_{i,j}^v = \mathcal{P}[\mathcal{I}_{i,j} f](V_i[r], \widehat{Q}_{i,1}[j_1], \widehat{Q}_{i,2}[j_2], \widehat{Q}_{i,3}[j_3])$

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- At triangle $T_k = \langle V_1, V_2, V_3 \rangle$
  - choose a (local) polynomial projector $\mathcal{J}_{k,j} f$
  - set $c_{k,j}^t = \mathcal{P}[\mathcal{J}_{k,j} f](Z_k[r], V_1[j_1], V_2[j_2], V_3[j_3])$

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- If $\mathcal{I}_{i,j} f$ and $\mathcal{J}_{k,j} f$ reproduce polynomials up to degree $d \leq 3r - 1$, then $\mathcal{Q} f$ reproduces such polynomials as well

  $\Rightarrow$ approximation order $d + 1$

- If $\mathcal{I}_{i,j} f$ and $\mathcal{J}_{k,j} f$ reproduce polynomials up to degree $3r - 1$, and if each of their supports belongs to a single triangle, then $\mathcal{Q} f$ is a projector in the spline space

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Conclusions

Normalized B-splines on PS-triangulations

- ☑ Local support
- ☑ Convex partition of unity
- ☑ Geometric construction:
  based on triangles that must contain a specific set of points
- ☑ Easy manipulation (two stages via Bernstein-Bézier form):
  stable evaluation, differentiation, integration
- ☑ Easy quasi-interpolation through blossoming
- ☒ Only particular combinations of polynomial degree/smoothness
- ☒ No recurrence relation

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