Polynomial identities

(1) Definition.

$$[\![]\!]^n : \mathbb{R} \to \mathbb{R} : \; x \mapsto x^n/n!$$

$$[\![]\!]^\alpha : \mathbb{R}^d \to \mathbb{R} : \; x \mapsto \prod_{j=1}^{d} [\![x(j)]\!]^{\alpha(j)}$$

(2) Multinomial identity.

$$[\![x + y + \cdots + z]\!]^\alpha = \sum_{\xi+\upsilon+\cdots+\zeta=\alpha} [\![x]\!]^\xi [\![y]\!]^\upsilon \cdots [\![z]\!]^\zeta$$

Proof: by induction on $|\alpha|$. □

(3) Taylor. *For any polynomial* p,

$$p(x+y) = \sum_\alpha [\![x]\!]^\alpha D^\alpha p(y).$$

Proof: For the particular polynomial $p = [\![]\!]^\beta$,

$$p(x+y) = \sum_\alpha [\![x]\!]^\alpha [\![y]\!]^{\beta-\alpha} = \sum_\alpha [\![x]\!]^\alpha (D^\alpha p)(y).$$

□

(4) Leibniz. *For any functions* $f, g, \ldots, h$ *and any scalar* s,

$$[\![sD]\!]^\alpha (fg\cdots h) = \sum_{\varphi+\gamma+\cdots+\eta=\alpha} [\![sD]\!]^\varphi f \; [\![sD]\!]^\gamma g \; \cdots [\![sD]\!]^\eta h$$

Proof: by induction on $|\alpha|$. □

(5) Leibniz-Hörmander (Hormander69: p. 10). *For any polynomial* p*, scalar* s*, and functions* f, g,

$$p(sD)(fg) = \sum_\beta \left(\left([\![D]\!]^\beta p\right)(sD)f\right) [\![sD]\!]^\beta g.$$

Proof:

$$\begin{aligned} p(sD)(fg) &= \sum_\alpha a(\alpha) [\![sD]\!]^\alpha (fg) \\ &= \sum_\alpha a(\alpha) \sum_\beta [\![sD]\!]^{\alpha-\beta} f \; [\![sD]\!]^\beta g \;=\; \sum_\beta \left(\left([\![D]\!]^\beta p\right)(sD)f\right) [\![sD]\!]^\beta g. \end{aligned}$$

□

Note. The identity is linear in p, f, g, hence verifiable by checking it just for pure powers.

(6) Convolution. *For any compactly supported* ϕ *and any* $p \in \Pi$,

$$\phi * p = p(\cdot - \mathrm{i}D)\widehat{\phi}(0) = \sum_{\gamma} [\![]\!]^{\gamma} D^{\gamma} p(-\mathrm{i}D)\widehat{\phi}(0) = \sum_{\gamma} D^{\gamma} p \, [\![- \mathrm{i}D]\!]^{\gamma} \widehat{\phi}(0).$$

Proof:

$$D^{\beta}\widehat{\phi} = \int \phi(y)(-\mathrm{i}y)^{\beta} \mathrm{e}^{-\mathrm{i}y()} \, \mathrm{d}y,$$

hence

$$(-\mathrm{i}D)^{\beta}\widehat{\phi}(0) = \int \phi(y)(-y)^{\beta} \, \mathrm{d}y.$$

So,

$$\begin{aligned}
\phi * [\![]\!]^{\alpha} &= \int \phi(\cdot - y)[\![y]\!]^{\alpha} \, \mathrm{d}y = \int \phi(y)[\![\cdot - y]\!]^{\alpha} \, \mathrm{d}y \\
&= \int \phi(y) \sum_{\beta} [\![]\!]^{\alpha-\beta} [\![- y]\!]^{\beta} \, \mathrm{d}y = \sum_{\beta} [\![]\!]^{\alpha-\beta} \int \phi(y)[\![- y]\!]^{\beta} \, \mathrm{d}y \, . \\
&= \sum_{\beta} [\![]\!]^{\alpha-\beta} [\![- \mathrm{i}D]\!]^{\beta} \widehat{\phi}(0) = [\![\cdot - \mathrm{i}D]\!]^{\alpha} \widehat{\phi}(0).
\end{aligned}$$

□

(This is an updated version of an appendix to Boor90c) 4jan09/24jul04/30aug99/...