175th Anniversary of University of Wisconsin-Madison: Games, complementarity and equilibria

Ever wondered why old buildings suddenly collapse, or why long hair moves in a certain pattern? Or how do earthquakes propagate, or how market power works? All of these can be explained mathematically using the concept of complementarity, a sort of either-or condition, or disjunctions as logicians like to call them.

The basic idea could be explained by a robot arm trying to move an object. Either there is a gap between the arm and the object, and the resultant force on the object is zero, or the arm is in contact with the object and the force could be positive. Mathematically, this amounts to force >=0 and gap >= 0 and one or other of these is zero, so called complementarity. In economics the notion is similar, excess supply Is nonnegative, prices are nonnegative and one or other of these will be zero when the system is balanced. Complementarity captures this notion well, often thought of as synonymous with equilibrium.

Generalizations to games abound, where players have individual objectives and choose strategies to optimize knowing that their outcomes are influenced by the strategies of other players. Complementarity encodes the simultaneous satisfaction of conditions of optimality for all players. The inception of the theory of games is widely attributed to John von Neumann and Oskar Morgenstern in the 1940s as a way to explain the workings of the stock market. Any good economist would tell you that Cournot and Bertrand had similar ideas in the mid to late1800s, and subsequent contributions by Zermelo, Borel and Steinhaus, among others occurred in the early 1900s. Gale, Kuhn and Nash (initially as graduate students) moved the field along during the 1950s. In terms of computation, Carlton Lemke, along with J. T. Howson, devised a clever algorithm for the bimatrix case (1964), and then followed that up with a pivoting algorithm for the linear complementarity problem (1965).

The University of Wisconsin has made significant impact on the field. In many practical settings, all players in the system are subjected to a set of common constraints. Think of players creating pollution subject to an environmental restriction. The resulting problem is a game with shared constraints, for which the usual fixed point analyses seem not to apply. While at Wisconsin, J. Ben Rosen (1965) devised an argument that found a way around this, and then showed the existence of a solution under certain checkable conditions, a result that is still heavily used today.

Along with his colleagues Olvi Mangasarian (incidentally the John von Neumann Professor of Computer Sciences at Wisconsin) and Klaus Ritter, Rosen organized the first symposium on nonlinear programming in 1970 and this generated huge interest in extensions of the optimization literature to these more general settings. The event was such a success that it led to subsequent meetings on nonlinear optimization, parallel optimization and complementarity over the ensuing decades.

The Army Mathematical Research Center (MRC) was also housed at Madison during that period, and a significant amount of research was devoted to methods for finding solutions to complementarity problems, with potential use in combat models and pricing (1993). Norman Josephy, a student of Stephen Robinson, devised extensions of Newton's method for this setting, while Mangasarian looked at reformulations of the problem as a system of equations (1976) as a solution approach, and independently adapting iterative methods from linear equations to a more general setup (1977). He also characterized those linear complementarity problems that are solvable by a single linear program (1976). Jong-Shi Pang, the co-author of two of the seminal books on complementarity and its variational extensions, spent a year at the MRC and the Mathematical Programming study of 1978 , Complementarity and Fixed Point Problems, contains some major milestone papers from Wisconsin in the development of methods for these problems.

The 1980s saw Robinson and his student Danny Ralph extend the work of Josephy to a more general setting and provide a framework for more general analysis. This work was implemented by Michael Ferris, Steven Dirkse, Youngdae Kim and Todd Munson as the PATH solver (1995), building on the work of Lemke and Lars Mathiesen, and this has become the standard method for solving large scale complementarity problems. New extensions of the nonlinear equation approaches using a semismooth approach have also proven useful in practice. At the same time, Stephen Wright was generalizing the interior point approaches of linear programming to the complementarity setting, leading to his elegant exposition of the underlying algorithmic principles (1997).

Other approaches have also emanated from Wisconsin, including the work of Bill Sandholm (2010). Applications to economics build on the computable general equilibrium setting, as popularized by Tom Rutherford. Ferris and Pang (1997) provide an authoritative survey of applications in science and engineering that remains topical more than 25 years on, and extensions to stochastic settings have been carried out by both Ferris and Robinson and their colleagues, along with related work on mechanism design by Shuchi Chawla. These concepts continue to impact energy and financial markets, as well as international trade.