$\mathbb{S}_n$FFT (supplement)

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1 Introduction

Harmonic analysis on $\mathbb{S}_n$ is defined via the notion of **representations**. A matrix valued function $\rho\colon \mathbb{S}_n \to \mathbb{C}^{d_\rho \times d_\rho}$ is said to be a d_ρ dimensional representation of the symmetric group if $\rho(\sigma_2)\rho(\sigma_1) = \rho(\sigma_2\sigma_1)$ for any pair of permutations $\sigma_1, \sigma_2 \in \mathbb{S}_n$. A representation ρ is said to be *reducible* if there exists a unitary basis transformation which simultaneously block diagonalizes each $\rho(\sigma)$ matrix into a direct sum of lower dimensional representations. If ρ is not reducible, then it is said to be *irreducible*. Irreducible representations or irreps are the elementary building blocks of all of $\mathbb{S}_n$'s representations. A complete set of inequivalent irreducible representations we denote by $\mathcal{R}$.

The Fourier transform of a function $f : \mathbb{S}_n \to \mathbb{C}$ is then defined as the sequence of matrices

$$\hat{f}(\rho) = \sum_{\sigma \in \mathbb{S}_n} f(\sigma)\rho(\sigma) \qquad \rho \in \mathcal{R}. \tag{1}$$

The inverse transform is

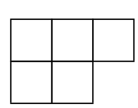
$$f(\sigma) = \frac{1}{n!} \sum_{\rho \in \mathcal{R}} d_\rho \operatorname{tr}\big[\hat{f}(\rho)\,\rho(\sigma)^{-1}\big] \qquad \sigma \in \mathbb{S}_n. \tag{2}$$

Much of the practical interest in Fourier transform can be attributed to various interesting properties of irreps, such as conjugacy and unitarity.

2 The irreducible representation of $\mathbb{S}_n$

There are several ways to construct irreducible representation of $\mathbb{S}_n$ [1]. One such representation is called Young's orthogonal representation (YOR). The YOR matrices are real and unitary and therefore orthogonal. To benefit from the computational advantages of orthogonal matrices, $\mathbb{S}_n$FFT uses YOR in its implementation. We provide a short review of the important concepts that are required for the construction of YOR.

Irreducible representation YOR of $\mathbb{S}_n$ is realized over conjugacy classes, namely partitions of n. A partition λ of n (denoted $\lambda \vdash n$) is a k-tuple $\lambda = (\lambda_1, ..., \lambda_k)$ of weakly decreasing positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = n$. A graphical sketch of a partition λ is a left-justified arrangement of empty boxes with λ_i boxes in i-th row. This is called a *Young diagram*, denoted F^λ. For example, the Young diagram corresponding to $\lambda = (3, 2)$ is



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A Young diagram bijectively filled with the integers $(1, 2, \cdots, n)$ produces a combinatorial object called a *Young tableau*. A tableau t is called *standard* if the assignment of numbers increases from left to right in each row and top to bottom in each column. The following are all the standard Young tableaux of shape $\lambda = (3, 2)$:

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} \tag{4}$$

Elements of $\mathbb{S}_n$ act on standard tableau in the obvious manner: permuting the numerals. In YOR, the irreducible representation corresponding to shape λ is a matrix of size $d_{\rho_\lambda} \times d_{\rho_\lambda}$. The columns (and rows) of ρ_λ are labeled by the distinct standard Young tableau of shape λ. The actual representation matrix corresponding to a given permutation $\sigma \in \mathbb{S}_n$ is constructed by first writing σ as a product of adjacent transpositions $\tau_1\tau_2 \cdots \tau_{n-1}$, which are of the form $\tau_i = (i, i+1)$, meaning that they interchange i with $i+1$ and leave everything else fixed. The matrix entries of Young's orthogonal representation corresponding to τ_i are calculated as follows:

$$\rho_\lambda(\tau_i)_{t,t'} = \begin{cases} (d_t(i, i+1))^{-1} & \text{if } t = t' \\ \sqrt{1 - (d_t(i, i+1))^{-2}} & \text{if } t' = \tau_i(t) \\ 0 & \text{otherwise,} \end{cases}$$

where d_t is the number of steps it takes to move i to $i + 1$[1]. Thus, $\rho_\lambda(\tau_i)$ is very sparse, which is critical for the FFT. Since adjacent transpositions generate the whole $\mathbb{S}_n$, the set of ρ_λ calculated for adjacent transpositions are sufficient to fully define the representation.

3 The Fast Fourier Transform on $\mathbb{S}_n$

$\mathbb{S}_n$FFT uses Clausen's celebrated fast Fourier transform algorithm [2], which systematically factors the elements of $\mathbb{S}_n$ into a product of *contiguous cycles*. A contiguous cycle $[\![i, j]\!] \in \mathbb{S}_n$ is a permutation of the form

$$[\![i, j]\!](k) = \begin{cases} k+1 & \text{for } k = i, i+1, .., j-1, \\ i & \text{for } k = j, \\ k & \text{otherwise} \end{cases} \qquad 1 \le i \le j \le n. \tag{5}$$

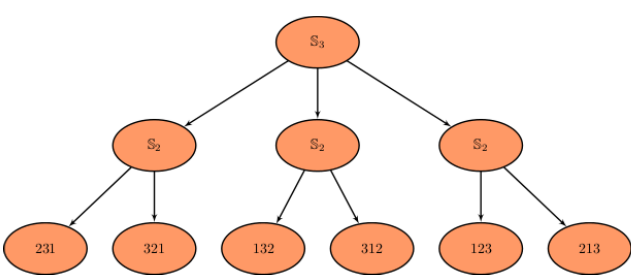

Figure 1: Left coset tree for $\mathbb{S}_3$ showing all members of $\mathbb{S}_3$ as leaves.

Two important properties of $[\![i, j]\!]$ are: *1*) each $[\![i, j]\!]$ factors into $j - i - 1$ adjacent transpositions; *2*) there is a unique factorization of any $\sigma \in \mathbb{S}_n$ into a product of contiguous cycles that is adapted to the chain of subgroups $\mathbb{S}_n \supset \mathbb{S}_{n-1} \supset \cdots \supset \mathbb{S}_1 = \texttt{id}$. Each subgroup creates a coset partition of $\mathbb{S}_n$, (Fig. 1). Clausen's FFT algorithm uses this nested chain of subgroups and proceeds by recursively breaking down Fourier transformation into smaller independent transforms. In particular, for a function $f : \mathbb{S}_n \to \mathbb{C}$, we define $f_i(\sigma') = f([\![i, n]\!]\sigma')$ for $\sigma' \in \mathbb{S}_{n-1}$ and $i = 1, 2, \cdots, n$, and let

$$\hat{f}_i(\rho_\mu) = \sum_{\sigma' \in \mathbb{S}_{n-1}} \rho_\mu(\sigma') f_i(\sigma') \qquad \mu \vdash n-1, \tag{6}$$

[1]North and east movements are taken positive, and south and west movements are taken negative.

then, up to the reordering of rows and columns, the fast Fourier transform computes $\widehat{f}$ from $\widehat{f}_1, \widehat{f}_2, \ldots, \widehat{f}_n$ in the form

$$\hat{f}(\rho_\lambda) = \sum_{i=1}^{n} \rho_\lambda([\![i,n]\!]) \bigoplus_{\mu \in \lambda\downarrow_{n-1}} \hat{f}_i(\rho_\mu) \qquad \lambda \vdash n. \tag{7}$$

Here $\lambda\downarrow_{n-1}$ denotes the set of irreps of $\mathbb{S}_{n-1}$ featured in the restriction of ρ_λ to $\mathbb{S}_{n-1}$. A more detailed technical description is available in [3].

References

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