The Complexity of Planar Boolean #CSP with Complex Weights

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Joint with:
Heng Guo (University of Wisconsin-Madison)

Quick Review: Previous Talks

- Xi Chen
    - $\#\mathrm{CSP}(\mathcal{F})$
    - Any domain size

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- Mingji Xia
  - $\text{Holant}^*(f)$ (symmetric arity 3)
  - Domain size 3

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  - Domain size 2 (Boolean domain)

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  - Holant($\mathcal{F}$)
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This talk:

- PI-#CSP($\mathcal{F}$)
- Domain size 2
- View PI-#CSP($\mathcal{F}$) in Holant framework

Quick Review: Symmetric signatures

$$\begin{aligned} \mathsf{OR}_2 &= [0, 1, 1] \\ \mathsf{AND}_3 &= [0, 0, 0, 1] \\ \text{EVEN-PARITY}_4 &= [1, 0, 1, 0, 1] \\ \mathsf{MAJORITY}_5 &= [0, 0, 0, 1, 1, 1] \\ (=_6) = \mathsf{EQUALITY}_6 &= [1, 0, 0, 0, 0, 0, 1] \end{aligned}$$

Quick Review: Symmetric signatures

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\mathrm{AND}_3 &= [0, 0, 0, 1] \\
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\mathrm{MAJORITY}_5 &= [0, 0, 0, 1, 1, 1] \\
(=_6) = \mathrm{EQUALITY}_6 &= [1, 0, 0, 0, 0, 0, 1]
\end{aligned}
$$

$$(=_n) = [1, 0, \ldots, 0, 1]^{\mathtt{T}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes n} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\otimes n}$$

Quick Review: Holographic transformation

- A transformation by the Hadamard matrix $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

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$$
\begin{aligned}
H^{\otimes n}(=_n) &= H^{\otimes n} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes n} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\otimes n} \right) \\
&= \left(H \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^{\otimes n} + \left(H \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^{\otimes n} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\otimes n} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{\otimes n} \\
&= [2, 0, 2, 0, 2, 0, 2, \ldots]^{\mathtt{T}} \qquad (n + 1 \text{ entries}) \\
&= 2 \cdot \text{EVEN-PARITY}_n
\end{aligned}
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NEW: Let $H\mathcal{F} = \widehat{\mathcal{F}}$

- A transformation by the Hadamard matrix $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

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NEW: Let $H\mathcal{F} = \widehat{\mathcal{F}}$
Note: $H\widehat{\mathcal{F}} = \mathcal{F}$ since $H\widehat{\mathcal{F}} = HH\mathcal{F} = 2\mathcal{F} = \mathcal{F}$

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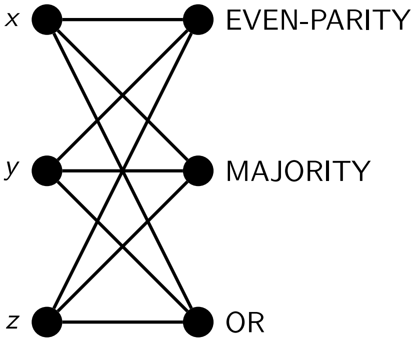

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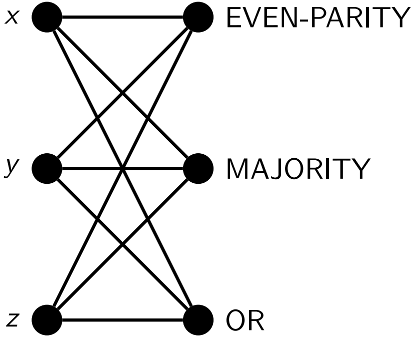

NOT planar, so **NOT** an instance of PI-#CSP($\{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_3\}$)

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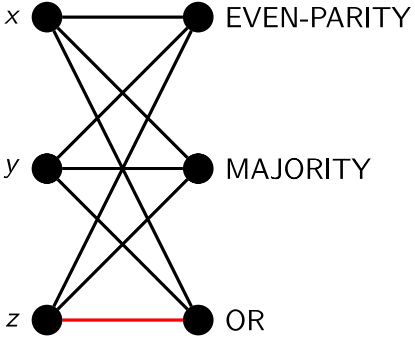

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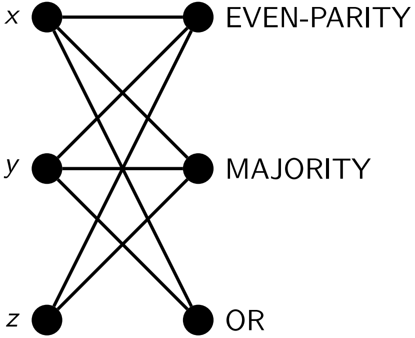
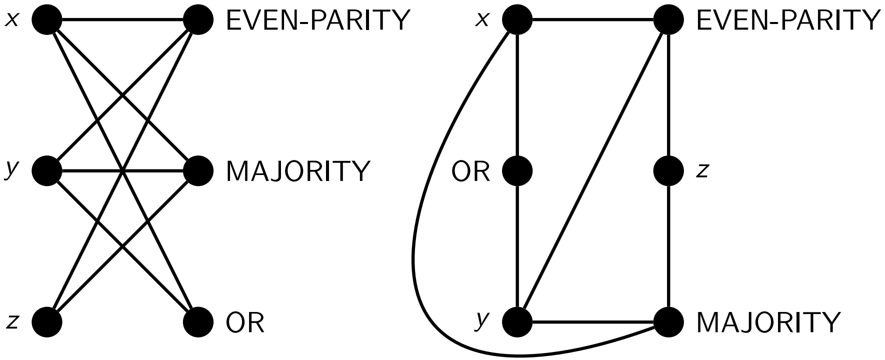


$$\text{EVEN-PARITY}(x, y, z) \wedge \text{MAJORITY}(x, y, z) \wedge \text{OR}(x, y)$$



VALID instance of PI-#CSP($\{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_2\}$)

#CSP($\mathcal{F}$)

- On input with (bipartite) constraint graph $G = (V, C, E)$, compute

$$\sum_{\sigma : V \to \{0,1\}} \prod_{c \in C} f_c \left(\sigma \mid_{N(c)} \right),$$

where $N(c)$ are the neighbors of c.

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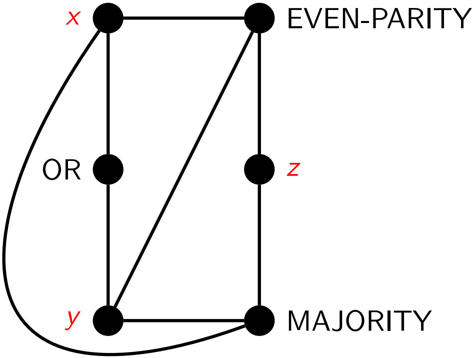
$$\#\mathrm{CSP}(\mathcal{F}) \equiv_T \mathrm{Holant}(\mathcal{EQ} \mid \mathcal{F}) \equiv_T \mathrm{Holant}(\mathcal{EQ} \cup \mathcal{F}),$$

where $\mathcal{EQ} = \{=_1, =_2, =_3, \dots\}$ is the set of equalities of all arities.

Example



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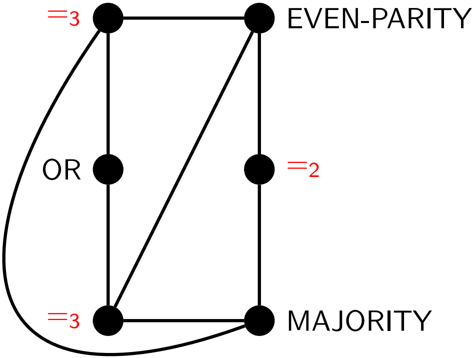

Some Signature Sets

Affine signatures $\mathscr{A} = \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, where

$$\mathscr{F}_1 = \left\{\lambda\left([1,0]^{\otimes k} + i^r[0,1]^{\otimes k}\right) \mid \lambda \in \mathbb{C}, k = 1,2,\ldots, r = 0,1,2,3\right\}$$

$$\mathscr{F}_2 = \left\{\lambda\left([1,1]^{\otimes k} + i^r[1,-1]^{\otimes k}\right) \mid \lambda \in \mathbb{C}, k = 1,2,\ldots, r = 0,1,2,3\right\}$$

$$\mathscr{F}_3 = \left\{\lambda\left([1,i]^{\otimes k} + i^r[1,-i]^{\otimes k}\right) \mid \lambda \in \mathbb{C}, k = 1,2,\ldots, r = 0,1,2,3\right\}.$$

Up to a scalar from $\mathbb{C}$:

1. $[1,0,\ldots,0,\pm 1]$; $(\mathscr{F}_1, r = 0,2)$
2. $[1,0,\ldots,0,\pm i]$; $(\mathscr{F}_1, r = 1,3)$
3. $[1,0,1,0,\ldots,0 \text{ or } 1]$; $(\mathscr{F}_2, r = 0)$
4. $[1,-i,1,-i,\ldots,(-i) \text{ or } 1]$; $(\mathscr{F}_2, r = 1)$
5. $[0,1,0,1,\ldots,0 \text{ or } 1]$; $(\mathscr{F}_2, r = 2)$
6. $[1,i,1,i,\ldots,i \text{ or } 1]$; $(\mathscr{F}_2, r = 3)$
7. $[1,0,-1,0,1,0,-1,0,\ldots,0 \text{ or } 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r = 0)$
8. $[1,1,-1,-1,1,1,-1,-1,\ldots,1 \text{ or } (-1)]$; $(\mathscr{F}_3, r = 1)$
9. $[0,1,0,-1,0,1,0,-1,\ldots,0 \text{ or } 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r = 2)$
10. $[1,-1,-1,1,1,-1,-1,1,\ldots,1 \text{ or } (-1)]$. $(\mathscr{F}_3, r = 3)$

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3. $[0, \alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n]$
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They satisfy

- Parity condition
- Geometric progression

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Example:

$$H\mathcal{E}\mathcal{Q} = \widehat{\mathcal{E}\mathcal{Q}} = \{2 \cdot \text{EVEN-PARITY}_n \mid n \in \mathbb{Z}^+\}$$

Main Result

Theorem

Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables.
Then PI-#CSP($\mathcal{F}$) *is* $\#\mathrm{P}$*-hard unless* $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{P}$, *or* $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, *in which case the problem is in* P.

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Theorem

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Secondary Result

Theorem

If f is a non-degenerate, symmetric, complex-valued signature of arity 4 in Boolean variables, then PI-Holant(f) is $\#\mathrm{P}$-hard unless f is

- $\mathscr{A}$*-transformable,*
- $\mathscr{P}$*-transformable,*
- *vanishing, or*
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Definition ($\mathcal{F}$-transformable)

A signature f is $\mathcal{F}$-transformable if there exists $T \in \mathbb{C}^{2\times 2}$ such that

- $f \in T\mathcal{F}$ and
- $=_2 T^{\otimes 2} \in \mathcal{F}$.

Previous Work: Planar Dichotomy Theorems

[Cai, Lu, Xia 10]

- Dichotomy for Pl-$\#$CSP($\mathcal{F}$) with **REAL** weights

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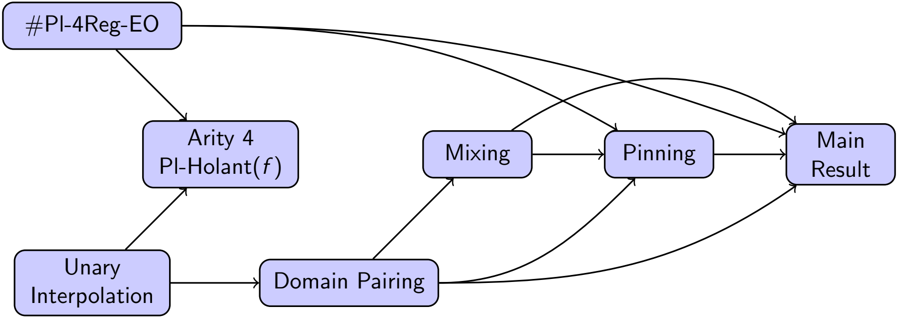
- Dichotomy for Pl-#CSP($\mathcal{F}$) with **REAL** weights
- Dichotomy for Pl-Holant(f) for arity 3 signature with complex weights

[Cai, Kowalczyk 10]

- Dichotomy for Pl-#CSP($[a, b, c]$) with complex weights

Proof Outline: Dependency Graph



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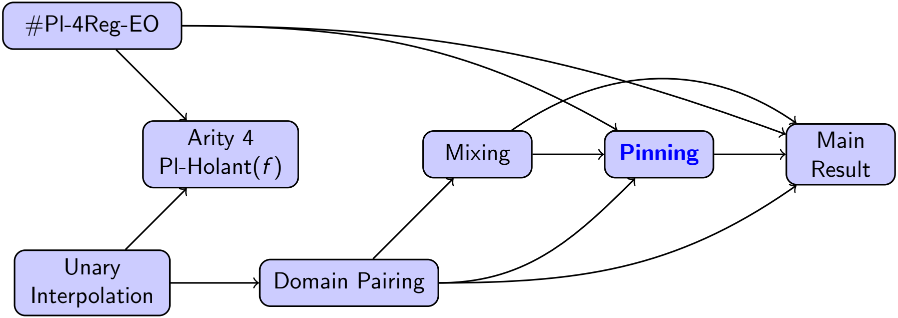

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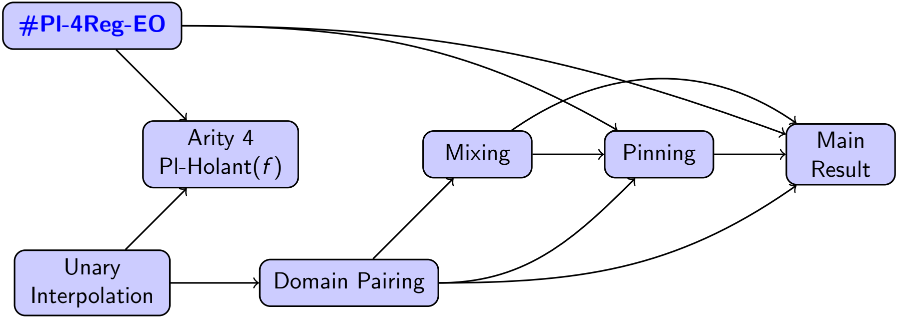

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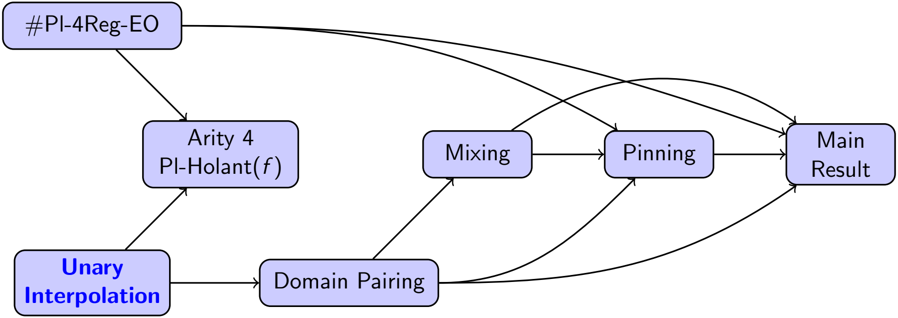

Pinning

Graph Homomorphism

- [Dyer, Greenhill 00]
- [Bulatov, Grohe 05]
- [Goldberg, Grohe, Jerrum, Thurley 10]
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Lemma

$$\#\mathrm{CSP}(\mathcal{F} \cup \{[1,0],[0,1]\}) \leq_T \#\mathrm{CSP}(\mathcal{F})$$

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PI-#CSP$(\widehat{\mathscr{M}} \cup \{[1,0],[0,1]\})$ is $\#\mathrm{P}$-hard but PI-#CSP$(\widehat{\mathscr{M}})$ is tractable

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Lemma

PI-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* $\#\mathrm{P}$*-hard (or in* P*)*

$$\Updownarrow$$

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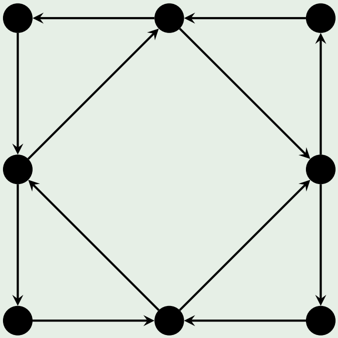
#PI-4Reg-EO: Eulerian Orientation

Definition

At each vertex in an Eulerian orientation of a graph,

in-degree equals out-degree.

Example



#PI-4Reg-EO: Theorem and Proof Overview

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is $\#\mathrm{P}$*-hard.*

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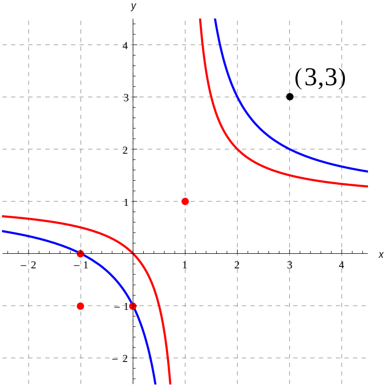
Proof.

Reduction from the evaluation of the Tutte polynomial at the point $(3, 3)$ for planar graphs:

$$\begin{aligned} \text{Pl-Tutte}(3,3) &\leq_T \quad \vdots \\ &\leq_T \#\text{Pl-4Reg-EO} \end{aligned}$$

Theorem (Vertigan 05)

For any $x, y \in \mathbb{C}$, the problem of computing the Tutte polynomial at (x, y) over planar graphs is $\#\mathrm{P}$-hard unless $(x-1)(y-1) \in \{1, 2\}$ or $(x, y) \in \{(1,1), (-1,-1), (j, j^2), (j^2, j)\}$, where $j = e^{2\pi i/3}$. In each of these exceptional cases, the computation can be done in polynomial time.

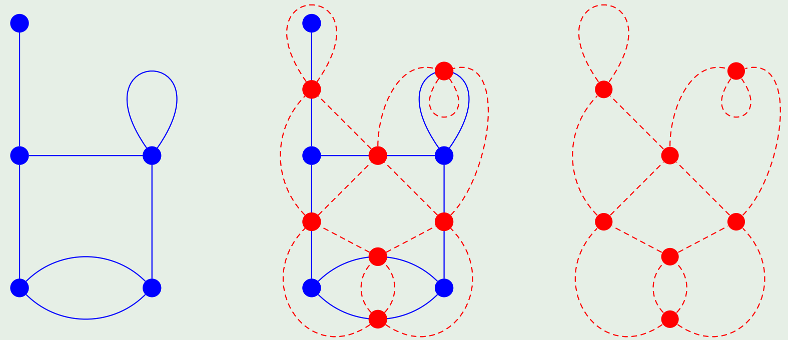


#PI-4Reg-EO: Medial Graph

Definition

For a connected plane graph G, its medial graph H has a vertex for each edge of G and two vertices in H are joined by an edge for each face of G in which their corresponding edges occur consecutively.

Example



Theorem (Las Vergnas 88)

Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{PI-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented "in, out, in, out" in cyclic order.

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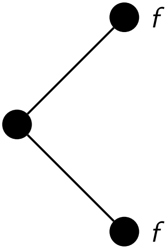
$(\neq_2) = [0,1,0]$ 

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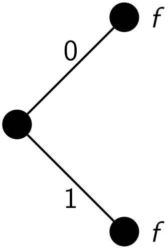

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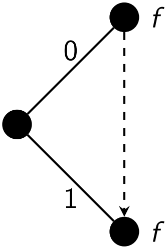
PI-Holant $([0,1,0] \mid f)$

$(\neq_2) = [0,1,0]$



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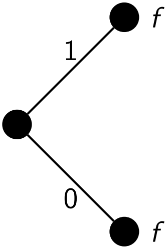

#PI-4Reg-EO: The Connection

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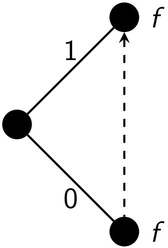

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Signature matrix:

- Let $f(w,x,y,z) = f^{wxyz}$ be an arity 4 signature
- Row index is (w,x), **BUT** the column index is (z,y) (order reversed)

$$M_f = \begin{bmatrix} f^{0000} & f^{0010} & f^{0001} & f^{0011} \\ f^{0100} & f^{0110} & f^{0101} & f^{0111} \\ f^{1000} & f^{1010} & f^{1001} & f^{1011} \\ f^{1100} & f^{1110} & f^{1101} & f^{1111} \end{bmatrix}$$

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- Let $f(w,x,y,z) = f^{wxyz}$ be an arity 4 signature
- Row index is (w,x), **BUT** the column index is (z,y) (order reversed)

$$M_f = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

#Pl-4Reg-EO: Proof Overview

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is $\#\mathrm{P}$-hard.

Proof.

$$\text{Pl-Tutte}(3,3) \equiv_T \text{Pl-Holant}\left([0,1,0] \mid \left[\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right]\right)$$

$$\leq_T \quad \vdots$$

$$\leq_T \#\text{Pl-4Reg-EO}$$

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Counting Eulerian Orientations for planar 4-regular graphs is $\#\mathrm{P}$*-hard.*

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$$\begin{aligned}
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&\leq_T \qquad \vdots \\
&\leq_T \text{Pl-Holant}([0,1,0] \mid \begin{bmatrix} 0&0&0&1\\ 0&1&1&0\\ 0&1&1&0\\ 1&0&0&0 \end{bmatrix} \quad) \\
&\equiv_T \#\text{Pl-4Reg-EO}
\end{aligned}$$

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Let $Z = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$.

#PI-4Reg-EO: Holographic Transformations

Let $Z = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$. Then

$$\begin{aligned} \text{PI-Holant}\left([0,1,0] \mid f\right) &\equiv_T \text{PI-Holant}\left([0,1,0](Z^{-1})^{\otimes 2} \mid Z^{\otimes 4} f\right) \\ &\equiv_T \text{PI-Holant}\left([1,0,1]/2 \mid 4\hat{f}\right) \\ &\equiv_T \text{PI-Holant}(\hat{f}), \end{aligned}$$

where

$$M_{\hat{f}} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Similarly,

$$\begin{aligned}
&\text{PI-Holant}\left([0,1,0] \mid [0,0,1,0,0]\right) \\
&\qquad \equiv_T \text{PI-Holant}\left([0,1,0](Z^{-1})^{\otimes 2} \mid Z^{\otimes 4}[0,0,1,0,0]\right) \\
&\qquad \equiv_T \text{PI-Holant}\left([1,0,1]/2 \mid 2[3,0,1,0,3]\right) \\
&\qquad \equiv_T \text{PI-Holant}([3,0,1,0,3]).
\end{aligned}$$

Theorem

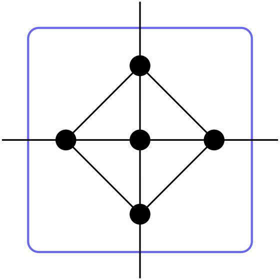
Counting Eulerian Orientations for planar 4-regular graphs is $\#\mathrm{P}$*-hard.*

Proof.

$$\begin{aligned}
\text{Pl-Tutte}(3,3) &\equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\equiv_T \text{Pl-Holant}\left(\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\right) \\
&\leq_T \qquad \vdots \\
&\leq_T \text{Pl-Holant}([3,0,1,0,3]) \\
&\equiv_T \text{Pl-Holant}([0,1,0] \mid [0,0,1,0,0]) \\
&\equiv_T \#\text{Pl-4Reg-EO}
\end{aligned}$$

#PI-4Reg-EO: Planar Tetrahedron Gadget

Assign $[3, 0, 1, 0, 3]$ to every vertex of this gadget...



...to get a signature $32\hat{g}$ with

$$M_{\hat{g}} = \frac{1}{2} \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}.$$

#PI-4Reg-EO: Proof Overview

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Counting Eulerian Orientations for planar 4-regular graphs is $\#\mathrm{P}$*-hard.*

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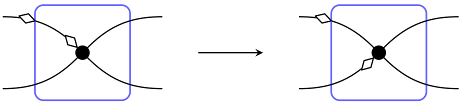
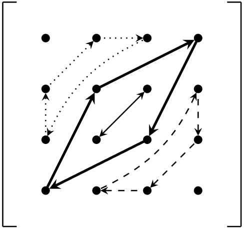
#PI-4Reg-EO: Rotationally Symmetric

$$M_{\hat{f}} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \qquad M_{\hat{g}} = \frac{1}{2} \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}$$



(a) A counterclockwise rotation.



(b) Movement of signature matrix entries under a counterclockwise rotation.

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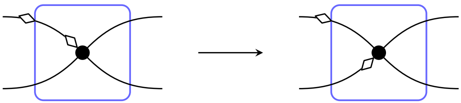

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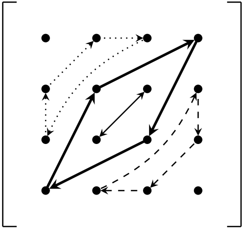

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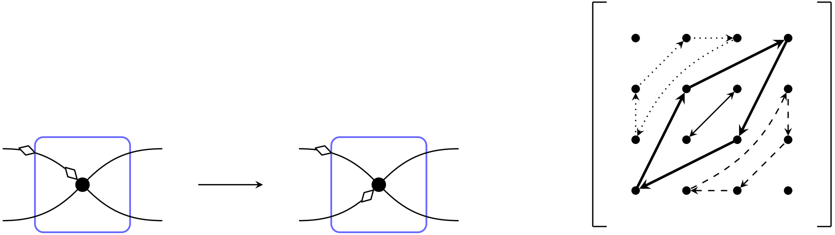

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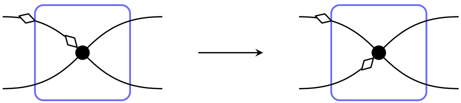

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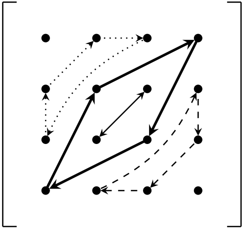
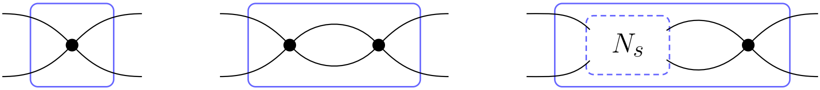


(b) Movement of signature matrix entries under a counterclockwise rotation.

Suppose that $\hat{f}$ appears n times in Ω of PI-Holant($\hat{f}$).
Construct instances Ω_s of Holant($\hat{g}$) indexed by $s \geq 1$.
Obtain Ω_s from Ω by replacing each $\hat{f}$ with N_s ($\hat{g}$ assigned to all vertices).



N_1 N_2 N_{s+1}

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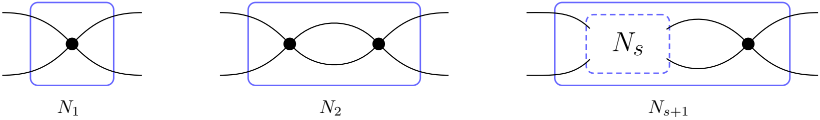

To obtain Ω_s from Ω,
we effectively replace $M_{\hat{f}}$ with $M_{N_s} = (M_{\hat{g}})^s$.

#PI-4Reg-EO: Interpolation

Let $T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$.

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$$M_{\hat{f}} = T \Lambda_{\hat{f}} T^{-1} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} T^{-1}$$

and

$$M_{\hat{g}} = T \Lambda_{\hat{g}} T^{-1} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix} T^{-1}.$$

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Follows from being both rotationally symmetric and complement invariant.

$$\Lambda_{\hat{f}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{\hat{g}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

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1. To obtain Ω_s from Ω,
   we first replace $M_{\hat{f}}$ with $T \Lambda_{\hat{f}} T^{-1}$. (Holant unchanged)

$$\Lambda_{\hat{f}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{\hat{g}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

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$$\Lambda_{\hat{f}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{\hat{g}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

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We only need to consider the assignments to $\Lambda_{\hat{f}}$ that assign

- 0000 j many times,
- 0110 or 1001 k many times, and
- 1111 ℓ many times.

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω.

$$\Lambda_{\hat{f}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{\hat{g}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

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2. Then we replace $T \Lambda_{\hat{f}} T^{-1}$ with $T (\Lambda_{\hat{g}})^s T^{-1}$.

We only need to consider the assignments to $\Lambda_{\hat{f}}$ that assign

- 0000 j many times,
- 0110 or 1001 k many times, and
- 1111 ℓ many times.

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω.

$$\Lambda_{\hat{f}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{\hat{g}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

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#PI-4Reg-EO: Interpolation

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$$\text{PI-Holant}_{\Omega} = \sum_{j+k+\ell=n} 3^{\ell} c_{jk\ell}$$

and

$$\text{PI-Holant}_{\Omega_s} = \sum_{j+k+\ell=n} (6^k 13^{\ell})^s c_{jk\ell}$$

is a full rank Vandermonde system (row index s, column index $c_{jk\ell}$).

#Pl-4Reg-EO: Proof Overview

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is $\#\mathrm{P}$*-hard.*

Proof.

$$
\begin{aligned}
\text{Pl-Tutte}(3,3) &\equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\equiv_T \text{Pl-Holant}\left(\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\right) \\
&\leq_T \text{Pl-Holant}\left(\frac{1}{2}\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}\right) \\
&\leq_T \text{Pl-Holant}([3,0,1,0,3]) \\
&\equiv_T \text{Pl-Holant}([0,1,0] \mid [0,0,1,0,0]) \\
&\equiv_T \#\text{Pl-4Reg-EO}
\end{aligned}
$$

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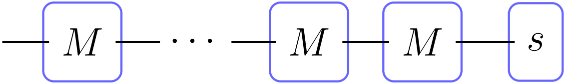
\end{aligned}$$

□

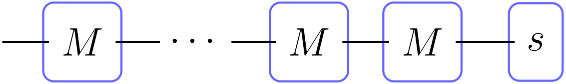
Major proof techniques:

1. Holographic transformation
2. Gadget construction
3. Interpolation



Recursive unary construction (M, s)

Unary Interpolation



Recursive unary construction (M, s)

Lemma (Vadhan 01, Cai-Lu-Xia 12, Cai-Lu-Xia 11)

Suppose $M \in \mathbb{C}^{2\times 2}$ *and* $s \in \mathbb{C}^{2\times 1}$.

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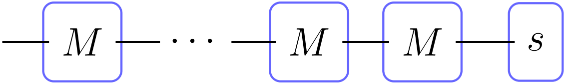

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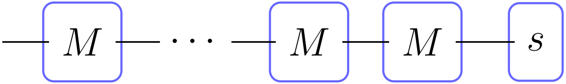
Suppose $M \in \mathbb{C}^{2\times 2}$ *and* $s \in \mathbb{C}^{2\times 1}$. *If*

then

the vectors in the set $V = \{M^k s\}_{k \geq 0}$ *are pairwise linearly independent.*



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Suppose $M \in \mathbb{C}^{2\times 2}$ *and* $s \in \mathbb{C}^{2\times 1}$*. If the following three conditions are satisfied,*

1. $\det(M) \neq 0$;
2. $\det([s \;\; Ms]) \neq 0$;
3. *the ratio of the eigenvalues of* M *is not a root of unity;*

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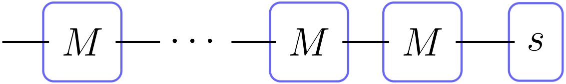

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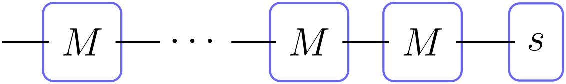

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Lemma (Our Result)

Suppose $M \in \mathbb{C}^{2\times 2}$ and $s \in \mathbb{C}^{2\times 1}$. If the following three conditions are satisfied,

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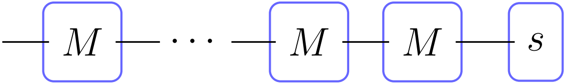

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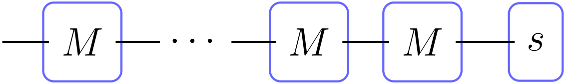

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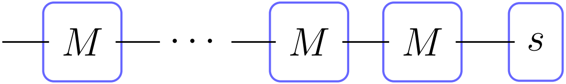
Suppose M has finite order modulo a scalar.



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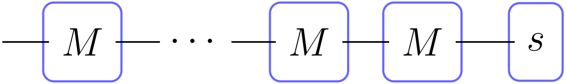
Suppose M has finite order modulo a scalar. Then we can construct M^{-1}.



Recursive unary construction (M, s)

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the vectors in the set $V = \{M^k s\}_{k \geq 0}$ *are pairwise linearly independent.*

Suppose M has finite order modulo a scalar. Then we can construct M^{-1}. This is called the anti-gadget technique [Cai, Kowalczyk, **W** 12].

Thank You

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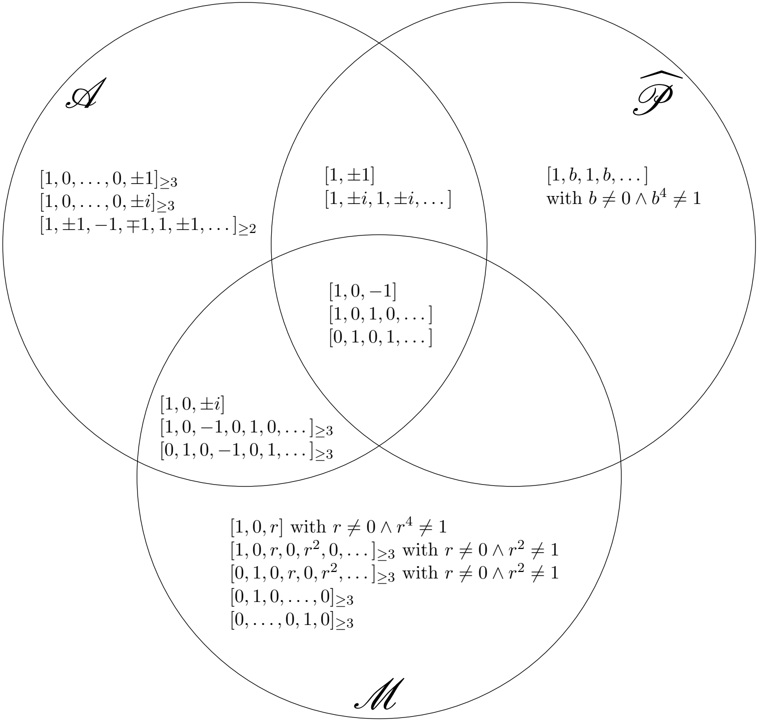
Paper and slides available on my website:
`www.cs.wisc.edu/~tdw`


$\mathscr{A}$
$\widehat{\mathscr{P}}$
$\mathscr{M}$
$[1, 0, \ldots, 0, \pm 1]_{\geq 3}$
$[1, 0, \ldots, 0, \pm i]_{\geq 3}$
$[1, \pm 1, -1, \mp 1, 1, \pm 1, \ldots]_{\geq 2}$
$[1, \pm 1]$
$[1, \pm i, 1, \pm i, \ldots]$
$[1, b, 1, b, \ldots]$
with $b \neq 0 \wedge b^4 \neq 1$
$[1, 0, -1]$
$[1, 0, 1, 0, \ldots]$
$[0, 1, 0, 1, \ldots]$
$[1, 0, \pm i]$
$[1, 0, -1, 0, 1, 0, \ldots]_{\geq 3}$
$[0, 1, 0, -1, 0, 1, \ldots]_{\geq 3}$
$[1, 0, r]$ with $r \neq 0 \wedge r^4 \neq 1$
$[1, 0, r, 0, r^2, 0, \ldots]_{\geq 3}$ with $r \neq 0 \wedge r^2 \neq 1$
$[0, 1, 0, r, 0, r^2, \ldots]_{\geq 3}$ with $r \neq 0 \wedge r^2 \neq 1$
$[0, 1, 0, \ldots, 0]_{\geq 3}$
$[0, \ldots, 0, 1, 0]_{\geq 3}$