The Complexity of Counting Problems

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Complexity Theory Review

In the beginning, there was SAT

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Output: "*Yes*" if there is a satisfying assignment
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Theorem (Cook '71, Levin '73)

SAT $\in$ NP-Complete.

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Example output: *No*

Theorem

3SAT $\in$ NP-Complete.

And 3SAT begat 1-in-3SAT and NAE-3SAT

Problem: 1-IN-3SAT
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Problem: NAE-3SAT
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Problem: 1-IN-3SAT
Input: A Boolean formula (in conjunctive normal form) such that each clause has exactly 3 literals.
Output: "*Yes*" if each clause has exactly 1 true literal
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1-IN-3SAT, NAE-3SAT $\in$ NP-Complete.

Then came the monotone versions

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A Boolean formula is monotone if the formula contains no negations.

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What is the complexity of...

- MON-3SAT?
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A Boolean formula is monotone if the formula contains no negations.

What is the complexity of...

- $\textsc{Mon}\text{-}3\mathrm{SAT} \in \mathrm{P}$
- $\textsc{Mon}\text{-}1\text{-}\textsc{in}\text{-}3\mathrm{SAT} \in$ NP-Complete
- $\textsc{Mon}\text{-}\mathrm{NAE}\text{-}3\mathrm{SAT} \in$ NP-Complete

What else is tractable?

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Input: A Boolean formula (in conjunctive normal form)
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$$(\overline{w} \lor \overline{y} \lor z) \land (w \lor \overline{x} \lor \overline{z}) \land (x \lor \overline{y} \lor \overline{z}) \land (\overline{w} \lor y \lor \overline{z}) \land (\overline{w} \lor \overline{x} \lor \overline{y})$$

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Theorem

HORN-SAT $\in$ P.

Constraint Satisfaction Problems

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Examples:

$$
\begin{array}{rll}
\mathrm{SAT} & \text{has} & \mathcal{F} = \{\mathsf{OR}_k \mid k \in \mathbb{N}\} \cup \{\mathsf{NOT}_2\} \\
\mathrm{3SAT} & \text{has} & \mathcal{F} = \{\mathsf{OR}_3, \mathsf{NOT}_2\} \\
\mathrm{1\text{-}\textsc{in}\text{-}3SAT} & \text{has} & \mathcal{F} = \{\mathsf{EXACTLY\text{-}ONE}_3, \mathsf{NOT}_2\} \\
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\\
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Theorem (Schaefer's dichotomy theorem '78)

*For any set of constraint functions $\mathcal{F}$,
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unless one of the following conditions holds,
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All constraints in $\mathcal{F}$...

1. *that are not constantly false are true when all its arguments are true;*
2. *that are not constantly false are true when all its arguments are false;*
3. *are equivalent to a conjunction of binary clauses;*
4. *are equivalent to a conjunction of Horn clauses;*
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6. *are equivalent to a conjunction of affine formula.*

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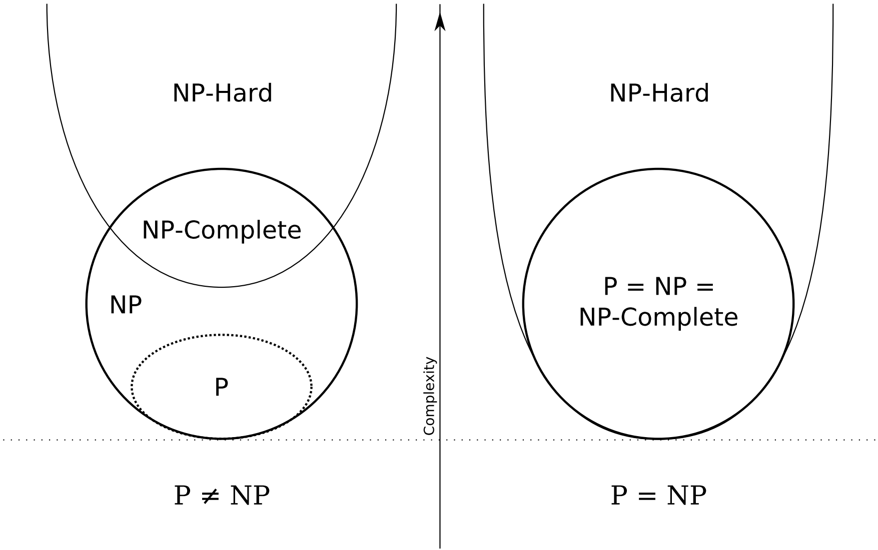
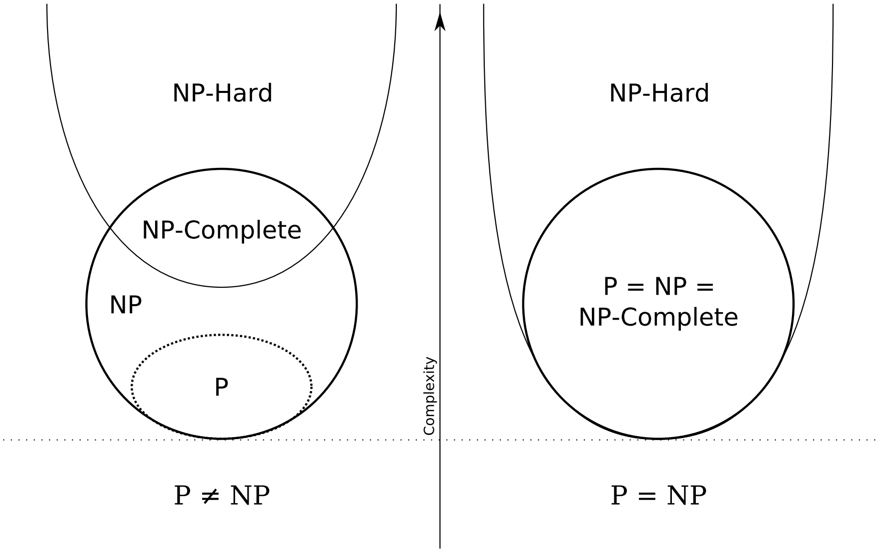
Theorem (Ladner's theorem '75)

If $\mathrm{P} \neq \mathrm{NP}$, then there exists problems in NP of intermediate complexity.



$P \neq NP$ $P = NP$

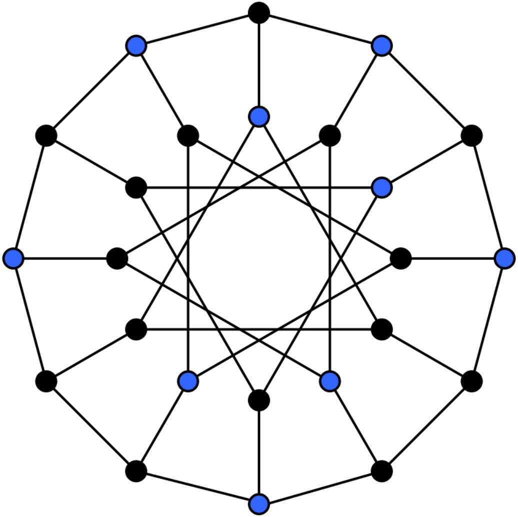


http://commons.wikimedia.org/wiki/File:P_np_np-complete_np-hard.svg

A Motivation for Counting

Independent Set

Want large independent set.



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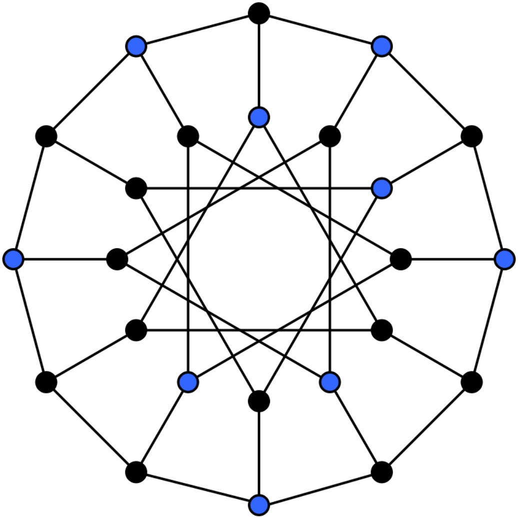

http://commons.wikimedia.org/wiki/File:Independent_set_graph.svg

Problem: IndependentSet
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INDEPENDENTSET $\in$ NP-Complete

Next question: how close to optimal can we get?

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Know as the partition function in statistical physics.

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Theorem (Sly,Sun '12)

For $d \geq 3$ and $\lambda > \lambda_c(d) = \frac{(d-1)^{d-1}}{(d-2)^d}$, unless $NP = RP$ there is no approximation algorithm for the partition function with $w(I) = \lambda^{|I|}$ on d-regular graphs.

Local Constraints

Definition

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex in the set.

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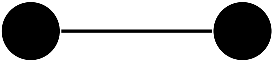

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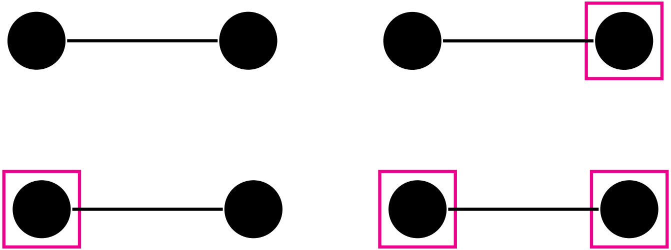

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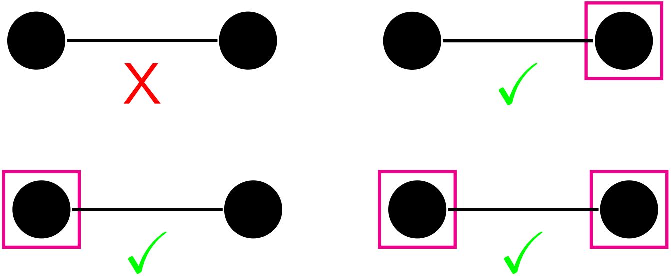
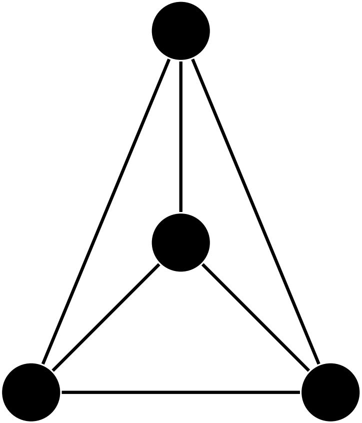


Systematic Approach to #VertexCover

- $G = (V, E)$



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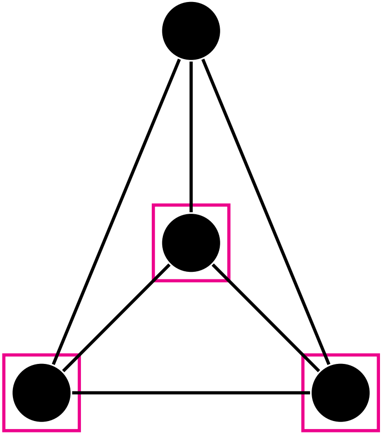
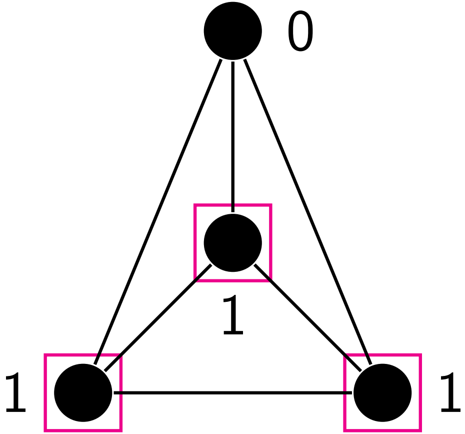


- $G = (V, E)$
- $\sigma : V \to \{0, 1\}$



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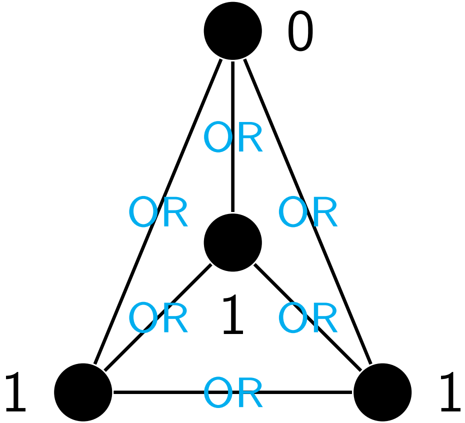

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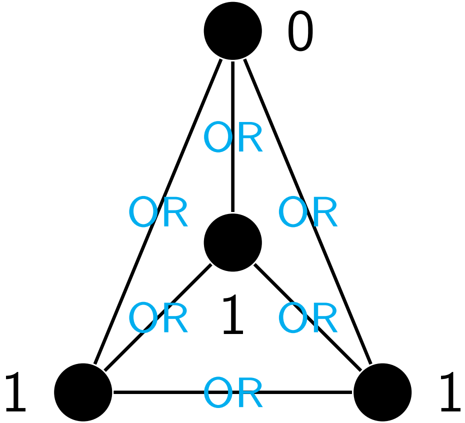

$$\prod_{(u,v)\in E} \mathrm{OR}(\sigma(u), \sigma(v)) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

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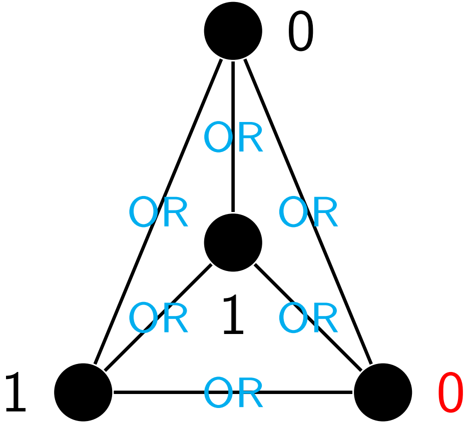

$$\prod_{(u,v) \in E} \mathrm{OR}(\sigma(u), \sigma(v)) = 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 0$$

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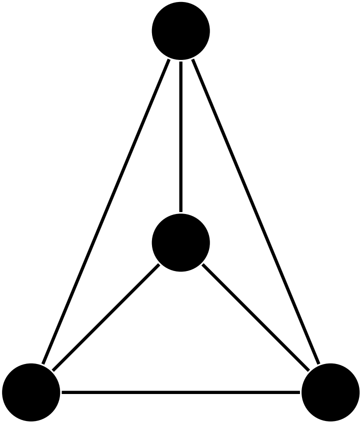

$$\#\textsc{VertexCover}(G) = \sum_{\sigma : V \to \{0,1\}} \prod_{(u,v) \in E} \mathrm{OR}(\sigma(u), \sigma(v))$$

Generalize

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Input		**Output**
p	q	$\mathrm{OR}(p, q)$
0	0	0
0	1	1
1	0	1
1	1	1

Generalize

$$\sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E} f(\sigma(u),\sigma(v))$$

Input		Output
p	q	$\mathrm{OR}(p,q)$
0	0	0
0	1	1
1	0	1
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Input		Output
p	q	$f(p,q)$
0	0	w
0	1	x
1	0	y
1	1	z

where $w, x, y, z \in \mathbb{C}$

Generalize

Partition Function: $Z(\cdot)$

$$Z(G) = \sum_{\sigma: V \to \{0,1\}} \prod_{(u,v) \in E} f(\sigma(u), \sigma(v))$$

Input		Output
p	q	$\mathrm{OR}(p, q)$
0	0	0
0	1	1
1	0	1
1	1	1

Input		Output
p	q	$f(p, q)$
0	0	w
0	1	x
1	0	y
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where $w, x, y, z \in \mathbb{C}$

Theorem (Cai, Kowalczyk, W '12)

Over 3-regular graphs G, the exact counting problem for any (binary) complex-weighted function f

$$Z(G) = \sum_{\sigma : V \to \{0,1\}} \prod_{(u,v) \in E} f(\sigma(u), \sigma(v))$$

is either computable in polynomial time or $\#\mathrm{P}$-hard.

Problem: HamiltonianCycle
Input: A graph G.
Output: "*Yes*" if G contains an Hamiltonian cycle
"*No*" otherwise.

Nonlocal Examples

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Confessions of a theorists:

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- Some proofs of this depending on definition of "local".
- Formally, just think of these as conjectures.

Symmetric Function

Definition

A function is symmetric if invariant under any permutation of its inputs.

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Examples:

$$\begin{aligned} \mathrm{OR}_2 &= [0, 1, 1] \\ \mathrm{AND}_3 &= [0, 0, 0, 1] \\ \text{EVEN-PARITY}_4 &= [1, 0, 1, 0, 1] \\ \mathrm{MAJORITY}_5 &= [0, 0, 0, 1, 1, 1] \\ (=_6) = \mathrm{EQUALITY}_6 &= [1, 0, 0, 0, 0, 0, 1] \end{aligned}$$

Constraint Graph for $\#\text{CSP}(\mathcal{F})$ Instance

$\mathcal{F} = \{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_3\}$

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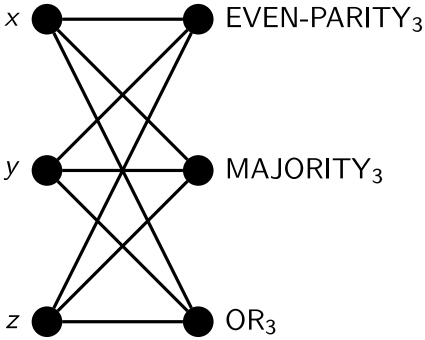

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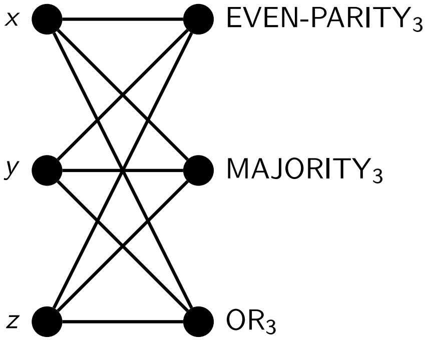

NOT planar, so **NOT** an instance of PI-#CSP($\{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_3\}$)

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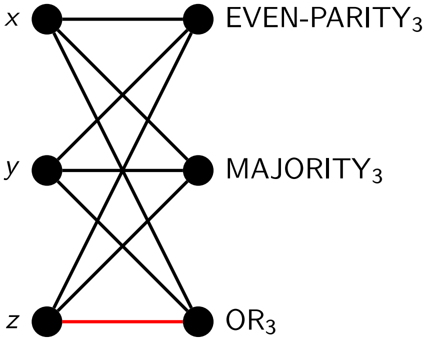

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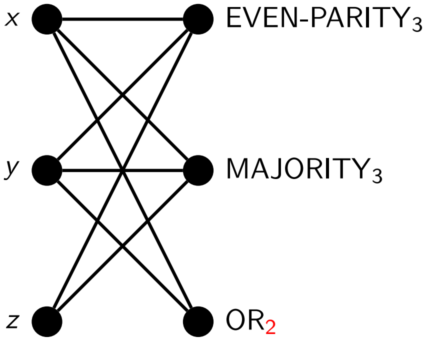

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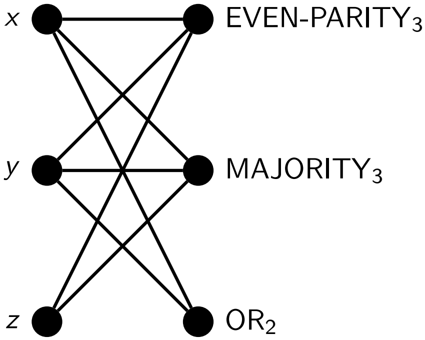

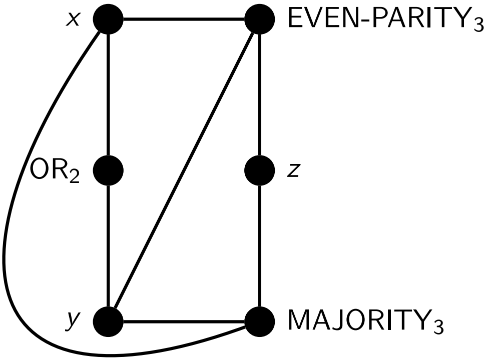

VALID instance of PI-$\#\text{CSP}(\{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_2\})$

More Counting Dichotomies

Theorem (Cai, Lu, Xia '09)

Let $\mathcal{F}$ be any set of complex-valued constraints in Boolean variables. Then $\#\mathsf{CSP}(\mathcal{F})$ is either $\#\mathrm{P}$-hard or computable in polynomial time.

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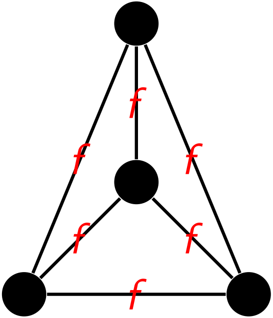
Theorem (Guo, W '13)

Let $\mathcal{F}$ be any set of symmetric, complex-valued constraints in Boolean variables. Then PI-$\#\text{CSP}(\mathcal{F})$ is either $\#\mathrm{P}$-hard or computable in polynomial time.

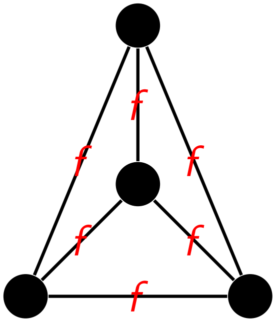
Definition of Holant Function

- Partition Function



$$\sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E} f(\sigma(u), \sigma(v))$$

- Partition Function
  - Assignments to vertices
  - Functions on edges



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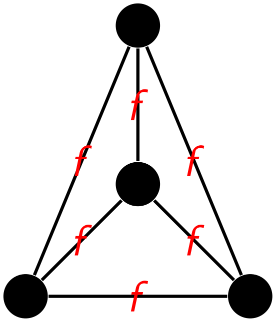

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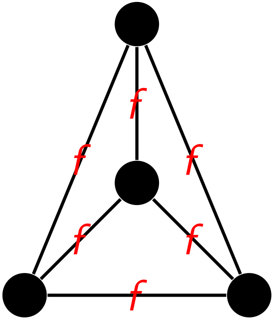

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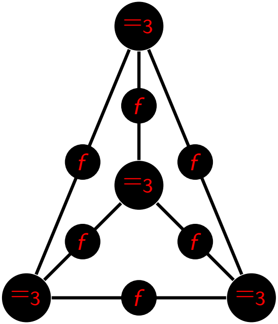

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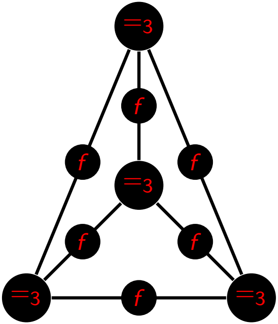

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- Degree 2 vertices take f.
- Degree 3 vertices take $=_3$.

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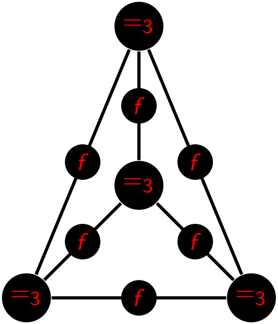

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- $\left.\begin{array}{l}\text{Holant}(\{=_2\} \mid \{\text{AT-MOST-ONE}\}) \\ \text{Holant}(\text{AT-MOST-ONE})\end{array}\right\}$ is $\#\textsc{Matching}$.

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- Holant($\{=_2\} \mid \{$AT-MOST-ONE$\}$), Holant(AT-MOST-ONE) } is #Matching.
- Holant($\{=_2\} \mid \{$EXACTLY-ONE$\}$), Holant(EXACTLY-ONE) } is #PerfectMatching.

Theorem (Cai, Guo, W '13)

Let $\mathcal{F}$ be any set of symmetric, complex-valued constraints in Boolean variables.
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A Proof Technique: Polynomial Interpolation

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Lemma

Given $n+1$ *distinct points* (x_i, y_i)*, there is a unique polynomial* $p(\cdot)$ *of degree at most* n *such that* $p(x_i) = y_i$.

Furthermore, the coefficients of p *can be computed in polynomial time.*

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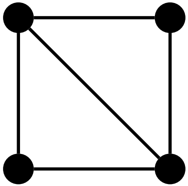

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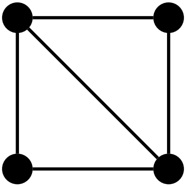

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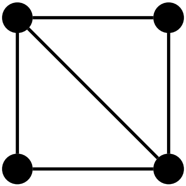

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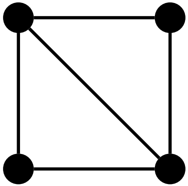

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$$\#\text{Matching}(G_\ell) = \sum_{k=0}^{n} m_k (\ell + 1)^k.$$

Thank You