Siegel's Theorem, Edge Coloring, and a Holant Dichotomy

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Carl L. Siegel

Theorem (Siegel's Theorem)

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Finiteness Theorems

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Theorem (Faltings' Theorem–Mordell Conjecture)

Any smooth algebraic curve of genus $g > 1$ *defined by a polynomial* $f(x, y) \in \mathbb{Z}[x, y]$ *has only finitely many rational solutions.*

Diophantine Equations with Enormous Solutions

Pell's Equation (genus 0)

$$x^2 - 61y^2 = 1$$

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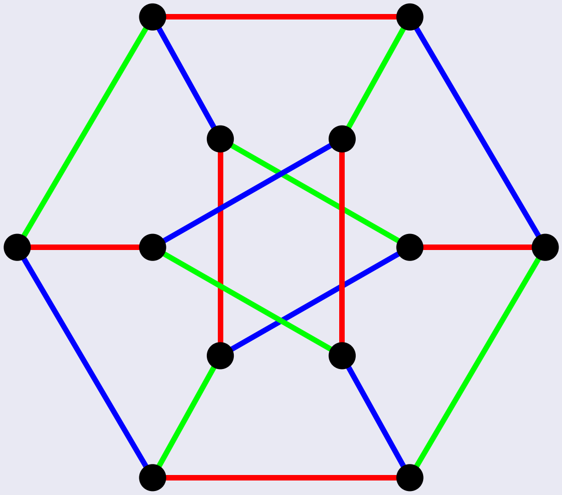
Edge Coloring

Definition



Edge Coloring–Decision Problem

Theorem (Vizing's Theorem)

Edge coloring using at most $\Delta(G) + 1$ colors exists.

Obvious lower bound is $\Delta(G)$.

Given G, deciding if $\Delta(G)$ colors suffice is NP-complete over

- 3-regular graphs [Holyer (1981)],
- k-regular graphs for $k \geq 3$ [Leven, Galil (1983)].

(No #P-hardness from these results.)

Edge Coloring–Decision Problem

1. Easy to show

$$k\text{-regular with bridge} \implies \text{no edge } k\text{-coloring exists.}$$

2. When planar 3-regular bridgeless, Tait (1880) proved

$$\text{edge 3-coloring exists} \iff \text{Four Color (Conjecture) Theorem.}$$

Therefore, for planar 3-regular graphs,

$$\text{edge 3-coloring exists} \iff \text{bridgeless.}$$

Edge Coloring–Counting Problem

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Input: A graph G.
Output: Number of edge colorings of G using at most κ colors.

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Theorem

$\#\kappa$-EdgeColoring is #P-hard over planar r-regular graphs for all $\kappa \geq r \geq 3$.

Trivially tractable when $\kappa \geq r \geq 3$ does not hold.

Proved in a framework of complexity dichotomy theorems in two cases:

1. $\kappa = r$, and
2. $\kappa > r$.

Three Frameworks for Counting Problems

1. Graph Homomorphisms
2. Constraint Satisfaction Problems (CSP)
3. Holant Problems

In each framework, there has been remarkable progress in the classification program of the complexity of counting problems.

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Let AD_3 denote the local constraint function

$$\text{AD}_3(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [\kappa] \text{ are distinct} \\ 0 & \text{otherwise.} \end{cases}$$

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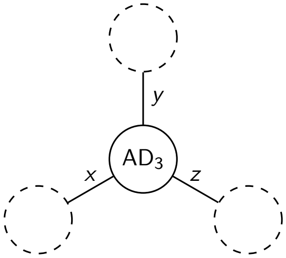
Place AD_3 at each vertex with incident edges x, y, z in a 3-regular graph G.



$\#\kappa$-EdgeColoring as a Holant Problem

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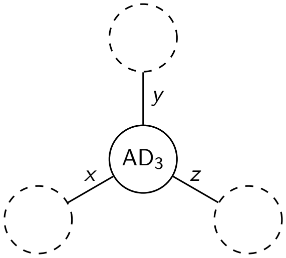

Then we evaluate the sum of product

$$\text{Holant}(G; AD_3) = \sum_{\sigma: E(G)\to[\kappa]} \prod_{v\in V(G)} AD_3\left(\sigma\mid_{E(v)}\right).$$

Clearly $\text{Holant}(G; AD_3)$ computes $\#\kappa$-EdgeColoring.
Same as contracting the corresponding tensor network.

In general, we consider all local constraint functions

$$f(x, y, z) = \begin{cases} a & \text{if } x = y = z \in [\kappa] \\ b & \text{if } |\{x, y, z\}| = 2 \\ c & \text{if } |\{x, y, z\}| = 3. \end{cases}$$

The Holant problem is to compute

$$\text{Holant}(G; f) = \sum_{\sigma : E(G) \to [\kappa]} \prod_{v \in V(G)} f\left(\sigma\,|_{E(v)}\right).$$

Denote f by $\langle a, b, c \rangle$.
Thus $\text{AD}_3 = \langle 0, 0, 1 \rangle$.

L. Lovász:
Operations with structures, Acta Math. Hung. 18 (1967), 321-328.
http://www.cs.elte.hu/~lovasz/hom-paper.html

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Let $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{\kappa \times \kappa}$ be a symmetric complex matrix.
The graph homomorphism problem is:
INPUT: An undirected graph $G = (V, E)$.
OUTPUT:

$$Z_{\mathbf{A}}(G) = \sum_{\xi : V \to [\kappa]} \prod_{(u,v) \in E} A_{\xi(u),\xi(v)}.$$

Examples of Graph Homomorphism

Let

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$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}.$$

Then $Z_{\mathbf{A}}(G)$ computes the number of vertex κ-colorings in G.

Theorem (Cai, Chen, Lu)

1. *For any symmetric complex-valued matrix* $\mathbf{A} \in \mathbb{C}^{\kappa \times \kappa}$, *the problem of computing* $Z_{\mathbf{A}}(\cdot)$ *is either in P or #P-hard.*
2. *Deciding whether* $Z_{\mathbf{A}}(\cdot)$ *is in P or #P-hard can be done in polynomial time (in the size of* $\mathbf{A}$*).*

SIAM J. Comput. 42(3): 924-1029 (2013) (106 pages)

Dichotomy Theorem for Graph Homomorphism

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Further generalized to all counting CSP.

Theorem (Cai, Chen)

1. *For any finite set* $\mathcal{F}$ *of complex-valued constraint functions over* $[\kappa]$, *the corresponding counting CSP problem* #CSP($\mathcal{F}$) *is in P or #P-hard.*

Unweighted decision version is open (Feder-Vardi Dichotomy Conjecture).

Dichotomy Theorem for $\text{Holant}(\cdot\,;\langle a, b, c\rangle)$

Theorem (Main Theorem)

1. *For any domain size* $\kappa \geq 3$ *and any* $a, b, c \in \mathbb{C}$, *the problem of computing* $\text{Holant}(\cdot\,;\langle a, b, c\rangle)$ *is in P or #P-hard, even when the input is restricted to planar graphs.*
2. *Deciding whether* $\text{Holant}(\cdot\,;\langle a, b, c\rangle)$ *is in P or #P-hard is very easy.*

Dichotomy Theorem for Holant($\cdot$; $\langle a, b, c\rangle$)

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Recall $\#\kappa$-EdgeColoring is the special case $\langle 0, 0, 1\rangle$.

Let's prove the theorem for this special case.

Nontrivial Examples of Tractable Holant Problems

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2. In general, Holant$(G; \langle \kappa^2 - 6\kappa + 4, -2(\kappa - 2), 4 \rangle)$ is in P.

3. On domain size $\kappa = 4$, Holant$(G; \langle -3 - 4i, 1, -1 + 2i \rangle)$ is in P.

Tutte Polynomial

Definition

The Tutte polynomial of an undirected graph G is

$$T(G;x,y) = \begin{cases} 1 & E(G) = \emptyset, \\ xT(G \setminus e; x, y) & e \in E(G) \text{ is a bridge}, \\ yT(G \setminus e; x, y) & e \in E(G) \text{ is a loop}, \\ T(G \setminus e; x, y) + T(G/e; x, y) & \text{otherwise}, \end{cases}$$

where $G \setminus e$ is the graph obtained by deleting e and G/e is the graph obtained by contracting e.

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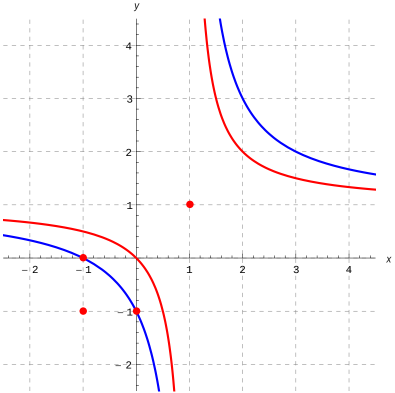
The chromatic polynomial is

$$\chi(G;\lambda) = (-1)^{|V|-1}\lambda\, \mathrm{T}(G; 1-\lambda, 0).$$

Tutte Polynomial Dichotomy

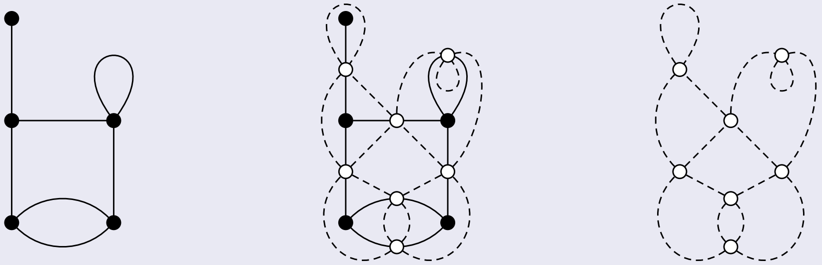
Theorem (Vertigan)

For any $x, y \in \mathbb{C}$, the problem of evaluating the Tutte polynomial at (x, y) over planar graphs is #P-hard unless $(x-1)(y-1) \in \{1, 2\}$ or $(x, y) \in \{(1,1), (-1,-1), (\omega, \omega^2), (\omega^2, \omega)\}$, where $\omega = e^{2\pi i/3}$. In each of these exceptional cases, the computation can be done in polynomial time.



Medial Graph

Definition

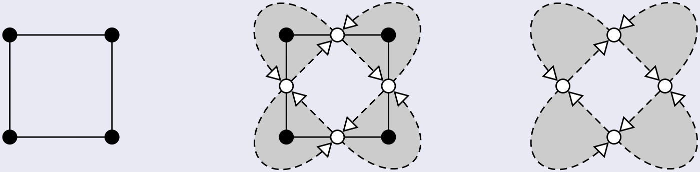


(a) (b) (c)

A plane graph (a), its medial graph (c), and the two graphs superimposed (b).

Directed Medial Graph

Definition



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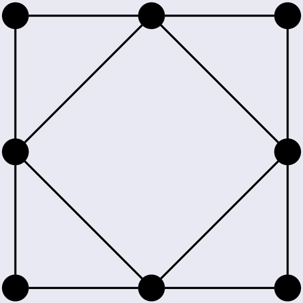

Eulerian Graphs and Eulerian Partitions

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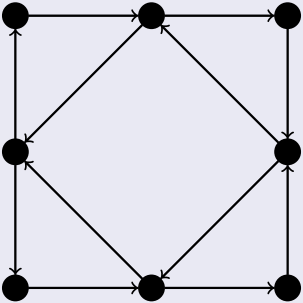
2. A digraph is Eulerian if “in degree” = “out degree” at every vertex.



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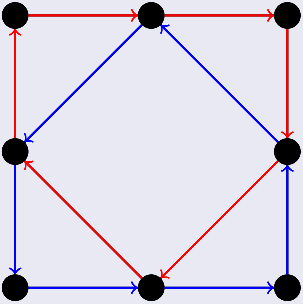
3. An Eulerian partition of an Eulerian digraph $\vec{G}$ is a partition of the edges of $\vec{G}$ such that each part induces an Eulerian digraph.
   Let $\pi_{\kappa}(\vec{G})$ be the set of Eulerian partitions of $\vec{G}$ into at most κ parts.



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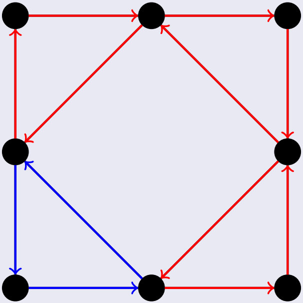
   Example with two monochromatic vertices (of degree 4).



Crucial Identity

Theorem (Ellis-Monaghan)

For a plane graph G,

$$\kappa\,\mathrm{T}(G;\kappa+1,\kappa+1)=\sum_{c\,\in\,\pi_\kappa(\vec{G}_m)} 2^{\mu(c)},$$

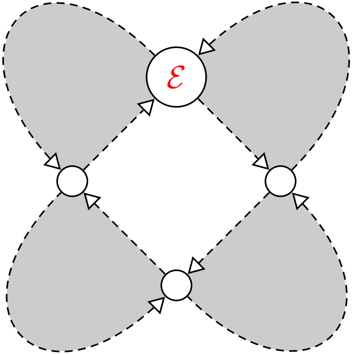
where $\mu(c)$ is the number of monochromatic vertices in c.

Then

$$\sum_{c \in \pi_\kappa(\vec{G}_m)} 2^{\mu(c)} = \text{Holant}(G_m; \mathcal{E}),$$

where

$$\mathcal{E}\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \begin{cases} 2 & \text{if } w = x = y = z \\ 1 & \text{if } w = x \neq y = z \\ 0 & \text{if } w = y \neq x = z \\ 1 & \text{if } w = z \neq x = y \\ 0 & \text{otherwise.} \end{cases}$$



Denote $\mathcal{E}$ by $\langle 2, 1, 0, 1, 0 \rangle$.

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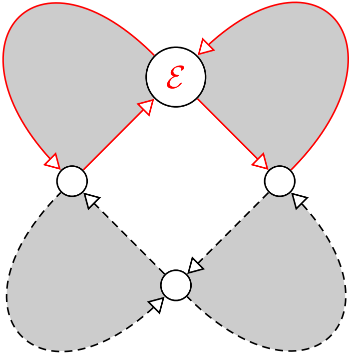

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Connection to Holant

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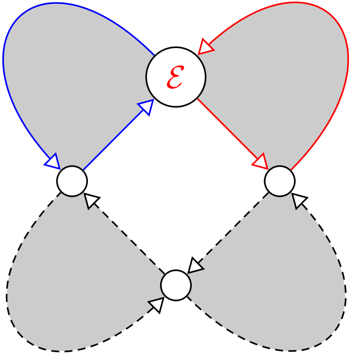

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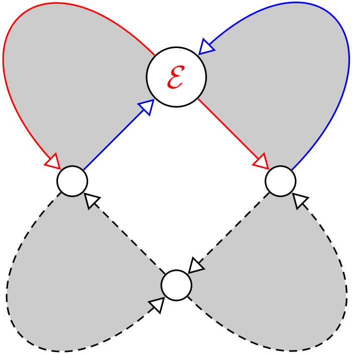

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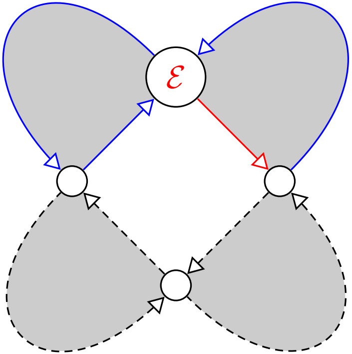

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#P-hardness of #κ-EdgeColoring

Theorem

#κ-EdgeColoring is #P-hard over planar κ-regular graphs for $\kappa \geq 3$.

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Proof for $\kappa = 3$.

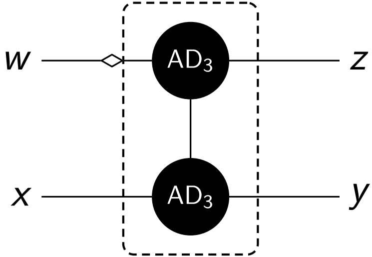
Reduce from Holant$(\cdot \,; \langle 2,1,0,1,0 \rangle)$ to Holant$(\cdot \,; \text{AD}_3)$ in two steps:

$$\text{Holant}(\cdot \,; \langle 2,1,0,1,0 \rangle) \leq_T \text{Holant}(\cdot \,; \langle 0,1,1,0,0 \rangle) \quad \text{(polynomial interpolation)}$$

$$\leq_T \text{Holant}(\cdot \,; \text{AD}_3) \quad \text{(gadget construction)}$$

Gadget Construction Step

$$\text{Holant}(G; \langle 0,1,1,0,0 \rangle) \leq_T \text{Holant}(G'; \text{AD}_3)$$



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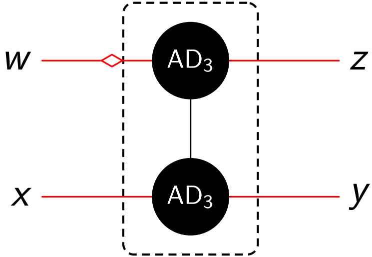

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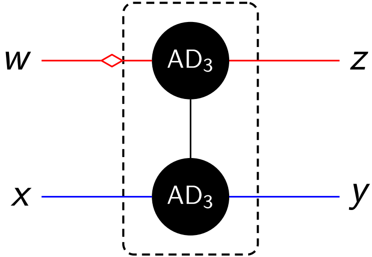

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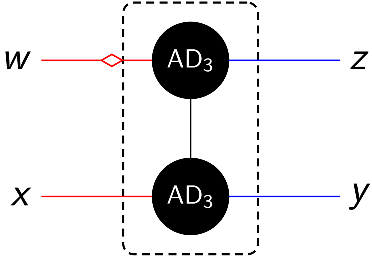

$$f\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \langle 0,1,1,0,0\rangle = \begin{cases} 0 & \text{if } w = x = y = z \\ 1 & \text{if } w = x \neq y = z \\ 1 & \text{if } w = y \neq x = z \\ 0 & \text{if } w = z \neq x = y \\ 0 & \text{otherwise.} \end{cases}$$

Gadget Construction Step

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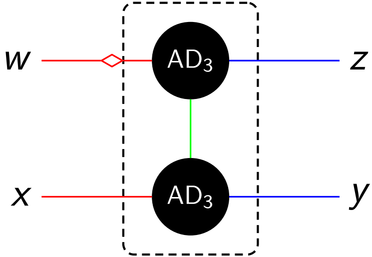

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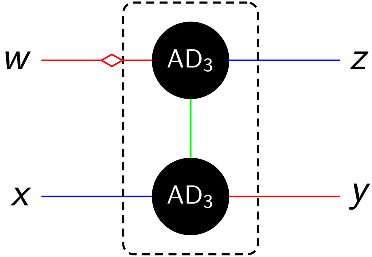

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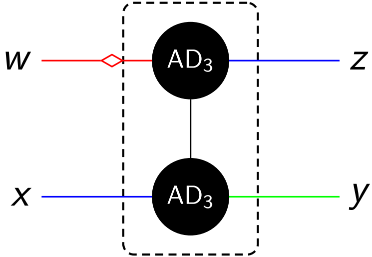

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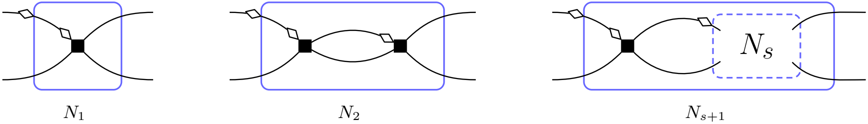
Polynomial Interpolation Step: Recursive Construction

$$\text{Holant}(G; \langle 2,1,0,1,0\rangle) \leq_T \text{Holant}(G_s; \langle 0,1,1,0,0\rangle)$$



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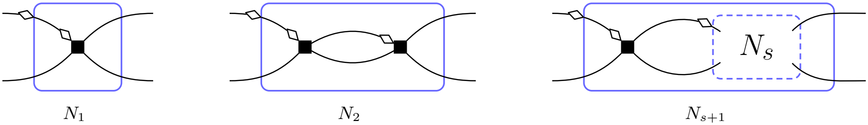

Vertices are assigned $\langle 0,1,1,0,0\rangle$.

Let f_s be the function corresponding to N_s. Then $f_s = M^s f_0$, where

$$M = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad f_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Obviously $f_1 = \langle 0,1,1,0,0\rangle$.

Polynomial Interpolation Step: Eigenvectors and Eigenvalues

Spectral decomposition $M = P\Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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$$f_{2s} = P\Lambda^{2s}P^{-1}f_0 = P \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}f_0 = \begin{bmatrix} \frac{x-1}{3} + 1 \\ \frac{x-1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

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(Side note: picking $s = 1$ so that $x = 4$ only works when $\kappa = 3$.)

Polynomial Interpolation Step: The Interpolation

$$\text{Holant}(G; \langle 2, 1, 0, 1, 0 \rangle) \leq_T \text{Holant}(G_s; \langle 0, 1, 1, 0, 0 \rangle)$$

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$$\begin{aligned}\text{Holant}(G; \langle 2,1,0,1,0\rangle) &= \text{Holant}(G; f(4)) \\ &\leq_T \text{Holant}(G; f(x)) \\ &\leq_T \text{Holant}(G_s; \langle 0,1,1,0,0\rangle)\end{aligned}$$

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Then $\text{Holant}(G_{2s}; \langle 0,1,1,0,0\rangle) = p(2^{2s})$.

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Using oracle for $\text{Holant}(\,\cdot\,; \langle 0,1,1,0,0\rangle)$, evaluate $p(x)$ at $n+1$ distinct points $x = 2^{2s}$ for $0 \leq s \leq n$.

By polynomial interpolation, efficiently compute the coefficients of $p(x)$. QED.

For all $a, b, c \in \mathbb{C}$,
want to show that $\text{Holant}(\,\cdot\,; \langle a, b, c\rangle)$ is in P or #P-hard.

Proof Outline for Dichotomy of Holant$(\cdot\,;\langle a,b,c\rangle)$

For all $a, b, c \in \mathbb{C}$,
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1. Attempt to construct a special arity 1 local constraint using $\langle a,b,c\rangle$.
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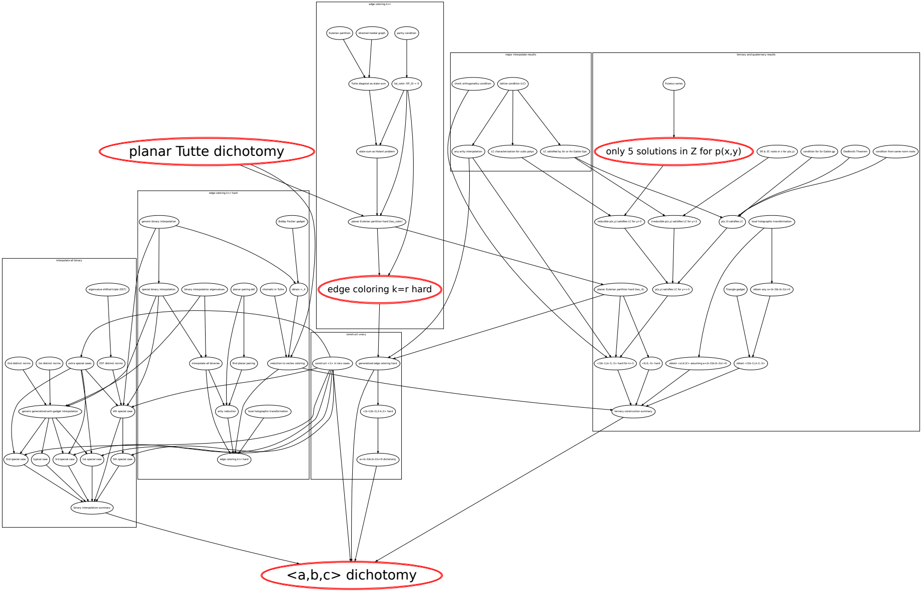
For some $a, b, c \in \mathbb{C}$, our attempts fail.

In those cases, we either

1. show the problem is in P or
2. prove #P-hardness without the help of additional signatures.


planar Tutte dichotomy
only 5 solutions in Z for p(x,y)
edge coloring k=r hard
<a,b,c> dichotomy

Interpolating Multivariate Polynomials

Let $p_d(X) \in \mathbb{Z}[X]$ be a polynomial of degree d.

$$\text{Can interpolate } p_d(X) \text{ from}$$
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Lattice Condition

Definition

We say that $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in \mathbb{C} - \{0\}$ satisfy the lattice condition if

$$\forall x \in \mathbb{Z}^\ell - \{\mathbf{0}\} \qquad \text{with} \qquad \sum_{i=1}^{\ell} x_i = 0,$$

we have

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Lemma

Let $p(x) \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 2$. If

1. *the Galois group of p over $\mathbb{Q}$ is S_n or A_n and*
2. *the roots of p do not all have the same complex norm,*

then the roots of p satisfy the lattice condition.

Interpolation Lemma

Lemma

If there exists an infinite sequence of $\mathcal{F}$-gates defined by an initial signature $s \in \mathbb{C}^{n \times 1}$ and a recurrence matrix $M \in \mathbb{C}^{n \times n}$ satisfying the following conditions,

1. *M is diagonalizable with n linearly independent eigenvectors;*
2. *s is not orthogonal to exactly ℓ of these linearly independent row eigenvectors of M with eigenvalues $\lambda_1, \ldots, \lambda_\ell$;*
3. *$\lambda_1, \ldots, \lambda_\ell$ satisfy the lattice condition;*

then

$$\text{Holant}(\,\cdot\,; \mathcal{F} \cup \{f\}) \leq_T \text{Holant}(\,\cdot\,; \mathcal{F})$$

for any signature f that is orthogonal to the $n - \ell$ of these linearly independent eigenvectors of M to which s is also orthogonal.

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Our proof applies this with $n = 9$ and $\ell = 5$.

$$\begin{bmatrix}
(\kappa-1)\left(\kappa^2+9\kappa-9\right) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-1)(2\kappa-3)(4\kappa-3) & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 3\kappa^3-28\kappa^2+60\kappa-36 & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-2)\left(\kappa^3-14\kappa^2+30\kappa-18\right) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(2\kappa-3)(4\kappa-3) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & 9\kappa^3-26\kappa^2+27\kappa-9 & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 2\left(\kappa^3-14\kappa^2+30\kappa-18\right) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-3)\left(\kappa^3-12\kappa^2+22\kappa-12\right) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
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\end{bmatrix}$$

The Recurrence Matrix

$$\begin{bmatrix}
(\kappa-1)\left(\kappa^2+9\kappa-9\right) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-1)(2\kappa-3)(4\kappa-3) & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
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Want to prove:

for all integers $y \geq 4$, the roots of $p(x,y)$ satisfy the lattice condition.

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Don't know how to prove this.

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For any integer $y \geq 1$, the polynomial $p(x, y)$ in x has three distinct real roots and two nonreal complex conjugate roots.

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Lemma

For any integer $y \geq 4$, if $p(x, y)$ is irreducible in $\mathbb{Q}[x]$, then the roots of $p(x, y)$ satisfy the lattice condition.

Proof.

Three distinct real roots do not have the same norm. An irreducible polynomial of prime degree n with exactly two nonreal roots has S_n as its Galois group over $\mathbb{Q}$. Hence the roots satisfy the lattice condition. □

What if Reducible?

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We know five integer solutions to $p(x, y) = 0$.
For these solutions, $p(x, y)$ is reducible:

$$p(x,y) = \begin{cases} (x-1)(x^4+x^3+2x^2-x+1) & y=-1 \\ x^2(x^3-x-2) & y=0 \\ (x+1)(x^4-x^3-2x^2-x+1) & y=1 \\ (x-1)(x^2-x-4)(x^2+2x+2) & y=2 \\ (x-3)(x^4+3x^3+2x^2-5x-9) & y=3. \end{cases}$$

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Lemma

Only integer solutions to $p(x,y) = 0$ *are*

$$(1,-1), (0,0), (-1,1), (1,2), (3,3).$$

If Reducible

Lemma

For any integer $y \geq 4$, if $p(x, y)$ is reducible in $\mathbb{Q}[x]$, then the roots of $p(x, y)$ satisfy the lattice condition.

Proof.

By previous lemma, no linear factor over $\mathbb{Z}$.
By Gauss' Lemma, no linear factor over $\mathbb{Q}$.
Then more Galois theory if $p(x, y)$ factors as a product of two irreducible polynomials of degrees 2 and 3. □

Effective Siegel's Theorem

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Consider

$$g_1(x, y) = y - x^2 \qquad \text{and} \qquad g_2(x, y) = \frac{y^2}{x} + y - x^2 + 1.$$

(This particular choice is due to Aaron Levin.)

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Then $g_1(a, b)$ and $g_2(a, b)$ are integers.

However, if $|a| > 16$, then either $g_1(a, b)$ or $g_2(a, b)$ is not an integer.

Puiseux Series

Puiseux series expansions for $p(x, y)$ are

$$y_1(x) = x^2 + 2x^{-1} + 2x^{-2} - 6x^{-4} - 18x^{-5} + O(x^{-6}),$$

$$y_2(x) = \quad x^{3/2} - \frac{1}{2}x + \frac{1}{8}x^{1/2} - \frac{65}{128}x^{-1/2} - x^{-1} - \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2}),$$

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Then

$$-1 < g_2\left(x, y_2^-(x)\right) \leq g_2\left(x, y_2(x)\right) < 0$$

for $x > 16$.

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for $x > 16$.

So for "large" x, $g_2(x, y_2(x))$ is not an integer.
Therefore, no "large" integral solutions.

Thank You

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Paper and slides available on my website:
`www.cs.wisc.edu/~tdw`