Siegel's Theorem, Edge Coloring, and a Holant Dichotomy

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Integer Solutions

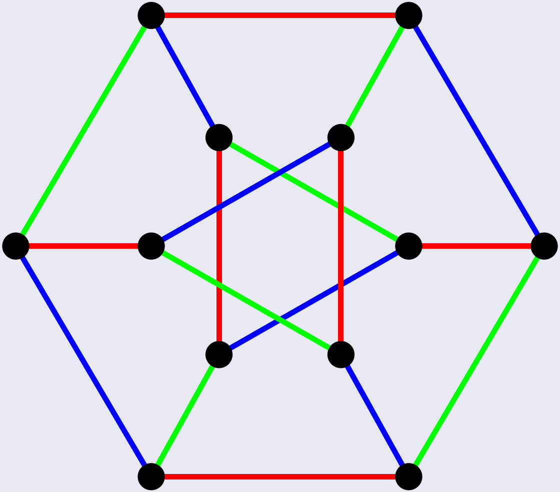
Theorem (Siegel's Theorem)

Any smooth algebraic curve of genus $g > 0$ defined by a polynomial $f(x, y) \in \mathbb{Z}[x, y]$ has only finitely many integer solutions.

Not effective.

Edge Coloring

Definition



Counting Edge Colorings

Problem: $\#\kappa$-EdgeColoring
Input: A graph G.
Output: Number of edge colorings of G using at most κ colors.

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Theorem

$\#\kappa$-EdgeColoring is #P-hard over planar r-regular graphs for all $\kappa \geq r \geq 3$.

Trivially tractable when $\kappa \geq r \geq 3$ does not hold.

Proved in two cases in the framework of Holant problems:

1. $\kappa = r$, and
2. $\kappa > r$.

#κ-EdgeColoring as a Holant Problem

Let AD_3 denote the local constraint function

$$\text{AD}_3(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [\kappa] \text{ are distinct} \\ 0 & \text{otherwise.} \end{cases}$$

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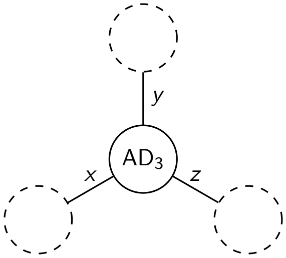
Place AD_3 at each vertex with incident edges x, y, z in a 3-regular graph G.



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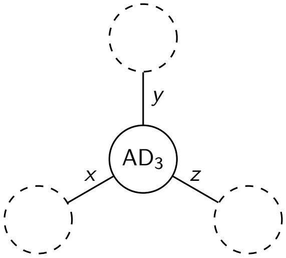


Then we evaluate the sum of product

$$\text{Holant}(G; \text{AD}_3) = \sum_{\sigma: E(G) \to [\kappa]} \prod_{v \in V(G)} \text{AD}_3\left(\sigma\mid_{E(v)}\right).$$

Clearly $\text{Holant}(G; \text{AD}_3)$ computes #κ-EdgeColoring.
Same as contracting the corresponding tensor network.

Holant Problems

In general, we consider all local constraint functions

$$f(x,y,z) = \begin{cases} a & \text{if } x = y = z \in [\kappa] \\ b & \text{if } |\{x,y,z\}| = 2 \\ c & \text{if } |\{x,y,z\}| = 3. \end{cases}$$

The Holant problem is to compute

$$\text{Holant}(G; f) = \sum_{\sigma: E(G) \to [\kappa]} \prod_{v \in V(G)} f\left(\sigma \mid_{E(v)}\right).$$

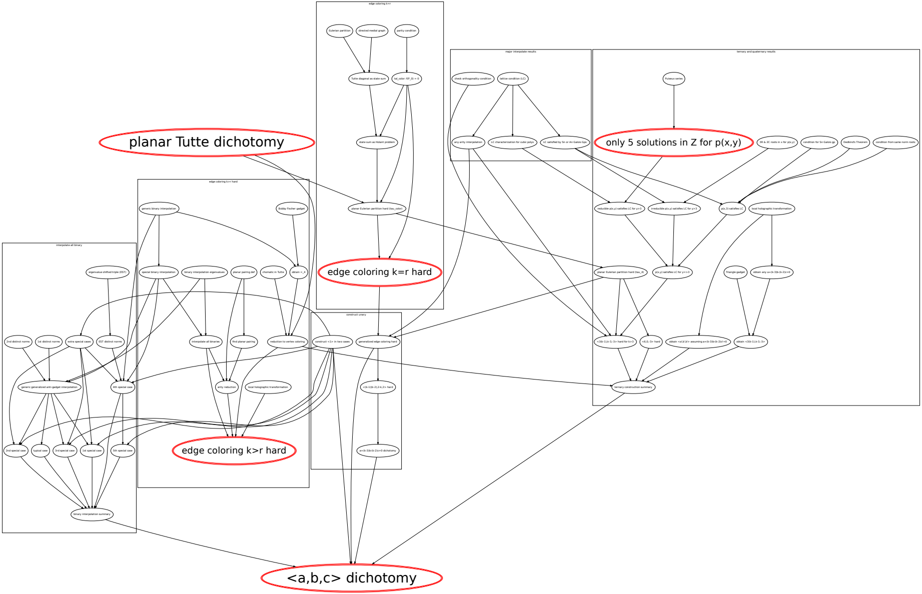
Denote f by $\langle a, b, c \rangle$.
Thus $\text{AD}_3 = \langle 0, 0, 1 \rangle$.

Dichotomy Theorem for $\text{Holant}(\cdot\,; \langle a, b, c\rangle)$

Theorem (Main Theorem)

For any domain size $\kappa \geq 3$ and any $a, b, c \in \mathbb{C}$, the problem of computing $\text{Holant}(\cdot\,; \langle a, b, c\rangle)$ is in P or #P-hard, even when the input is restricted to planar graphs.

Recall $\#\kappa$-EdgeColoring is the special case $\text{AD}_3 = \langle 0, 0, 1\rangle$.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edge coloring k=r hard
only 5 solutions in Z for p(x,y)
edge coloring k>r hard
<a,b,c> dichotomy
edge coloring k=r
major interpolate results
ternary and quaternary results
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interpolate all binary
construct unary
Eulerian partition
directed medial graph
parity condition
Tutte diagonal as state sum
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Puiseux series
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local holographic transformation
Triangle gadget
ternary construction summary
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chromatic in Tutte
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binary interpolation summary
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We know five integer solutions to $p(x,y) = 0$.

$$p(x,y) = \begin{cases} (x-1)(x^4+x^3+2x^2-x+1) & y=-1 \\ x^2(x^3-x-2) & y=0 \\ (x+1)(x^4-x^3-2x^2-x+1) & y=1 \\ (x-1)(x^2-x-4)(x^2+2x+2) & y=2 \\ (x-3)(x^4+3x^3+2x^2-5x-9) & y=3. \end{cases}$$

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Lemma

Only integer solutions to $p(x,y) = 0$ *are*

$$(1,-1),(0,0),(-1,1),(1,2),(3,3).$$

Effective Siegel's Theorem

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Consider

$$g_1(x, y) = y - x^2 \qquad \text{and} \qquad g_2(x, y) = \frac{y^2}{x} + y - x^2 + 1.$$

(This particular choice is due to Aaron Levin.)

Then $g_1(a, b)$ and $g_2(a, b)$ are integers.

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Then $g_1(a, b)$ and $g_2(a, b)$ are integers.

However, if $|a| > 16$, then either $g_1(a, b)$ or $g_2(a, b)$ is not an integer.

Puiseux Series

Puiseux series expansions for $p(x, y)$ are

$$y_1(x) = x^2 + 2x^{-1} + 2x^{-2} - 6x^{-4} - 18x^{-5} + O(x^{-6}),$$

$$y_2(x) = \quad x^{3/2} - \frac{1}{2}x + \frac{1}{8}x^{1/2} - \frac{65}{128}x^{-1/2} - x^{-1} - \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2}),$$

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$$-1 < g_2\left(x, y_2^-(x)\right) \leq g_2\left(x, y_2(x)\right) < 0$$

for $x > 16$.

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So for “large” x, $g_2(x, y_2(x))$ is not an integer.
Therefore, no “large” integral solutions.

Thank You

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Paper and slides available on my website:
`www.cs.wisc.edu/~tdw`