Siegel's theorem, edge coloring, and a holant dichotomy

Tyson Williams
(University of Wisconsin-Madison)

Joint with:
Jin-Yi Cai and Heng Guo
(University of Wisconsin-Madison)

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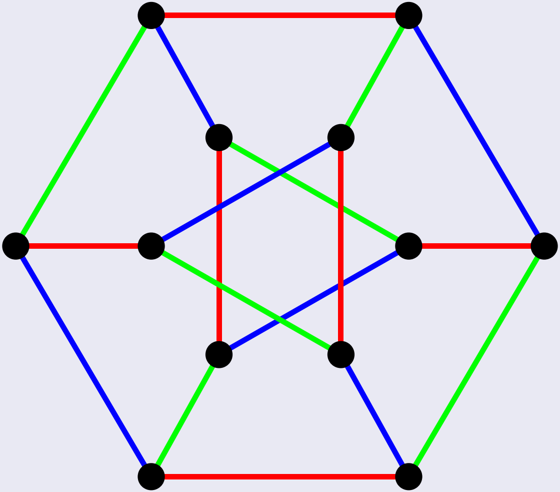
Edge Coloring

Definition



Counting Edge Colorings

Problem: $\#\kappa$-EdgeColoring
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Tractable when $\kappa \geq r \geq 3$ does not hold.

- If $\kappa < r$, then no edge colorings.
- If $r < 3$, then only trivial graphs (paths and cycles)

Parallel edges allowed (and necessary when $r > 5$).

Proved in the framework of Holant problems in two cases:

1. $\kappa = r$, and
2. $\kappa > r$.

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independent sets, vertex covers, edge covers, vertex colorings, edge colorings, matchings, perfect matchings, Eulerian orientations, and cycle covers.

NON-examples: Hamiltonian cycles and spanning trees.

NOT local.

Abundance of Holant Problems

Equivalent to:

- counting read-twice constraint satisfaction problems,
- contraction of tensor networks, and
- partition function of graphical models (in Forney normal form).

Generalizes:

- simulating quantum circuits,
- counting graph homomorphisms,
- all manner of partition functions including
  - Ising model,
  - Potts model,
  - edge-coloring model.

#κ-EdgeColoring as a Holant Problem

Let AD_3 denote the local constraint function

$$\mathrm{AD}_3(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [\kappa] \text{ are distinct} \\ 0 & \text{otherwise.} \end{cases}$$

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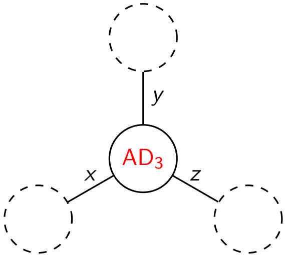
Place AD_3 at each vertex with incident edges x, y, z in a 3-regular graph G.



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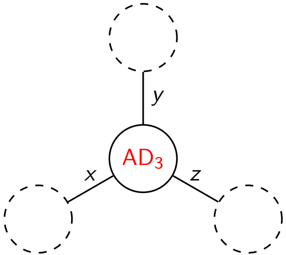

Then we evaluate the sum of product

$$\mathrm{Holant}(G; \mathrm{AD}_3) = \sum_{\sigma : E(G) \to [\kappa]} \prod_{v \in V(G)} \mathrm{AD}_3 \left(\sigma \mid_{E(v)} \right).$$

Clearly $\mathrm{Holant}(G; \mathrm{AD}_3)$ computes #κ-EdgeColoring.

In general, we consider all local constraint functions

$$f(x,y,z) = \begin{cases} a & \text{if } x = y = z \qquad \text{(all equal)} \\ b & \text{otherwise} \\ c & \text{if } x \neq y \neq z \neq x \qquad \text{(all distinct).} \end{cases}$$

The Holant problem is to compute

$$\text{Holant}_{\kappa}(G; f) = \sum_{\sigma : E(G) \to [\kappa]} \prod_{v \in V(G)} f\left(\sigma \mid_{E(v)}\right).$$

Denote f by $\langle a, b, c \rangle$.
Thus $\text{AD}_3 = \langle 0, 0, 1 \rangle$.

Theorem (Main Theorem)

For any $\kappa \geq 3$ *and any* $a, b, c \in \mathbb{C}$,
the problem of computing $\text{Holant}_\kappa(-; \langle a, b, c\rangle)$ *is in P or #P-hard,*
even when the input is restricted to planar graphs.

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Recall $\#\kappa$-EdgeColoring is the special case $\langle a, b, c\rangle = \langle 0, 0, 1\rangle$.

Let's prove the theorem for $\kappa = 3$ and $\langle a, b, c\rangle = \langle 0, 0, 1\rangle$.

Nontrivial Examples of Tractable Holant Problems

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   $$\langle -5, -2, 4 \rangle = \left[(1, -2, -2)^{\otimes 3} + (-2, 1, -2)^{\otimes 3} + (-2, -2, 1)^{\otimes 3} \right],$$

   do a holographic transformation by the orthogonal matrix
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2. In general,
   $\text{Holant}_\kappa(G; \langle \kappa^2 - 6\kappa + 4, -2(\kappa - 2), 4 \rangle)$ is in P.

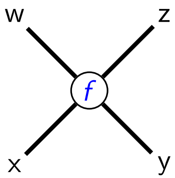
3. On domain size $\kappa = 4$,
   $\text{Holant}_4(G; \langle -3 - 4i, 1, -1 + 2i \rangle)$ is in P.

Hardness of $\text{Holant}_3(-; \text{AD}_3)$ proved by the following reduction chain:

$$\begin{aligned} \#\text{P} &\leq_T \text{Holant}_3(-; \langle 2, 1, 0, 1, 0 \rangle) \\ &\leq_T \text{Holant}_3(-; \langle 0, 1, 1, 0, 0 \rangle) \\ &\leq_T \text{Holant}_3(-; \text{AD}_3) \end{aligned}$$

$\langle a, b, c, d, e \rangle$ denotes an arity-4 function f

$$f\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \begin{cases} a & \text{if } w = x = y = z \\ b & \text{if } w = x \neq y = z \\ c & \text{if } w = y \neq x = z \\ d & \text{if } w = z \neq x = y \\ e & \text{otherwise.} \end{cases}$$



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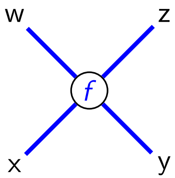

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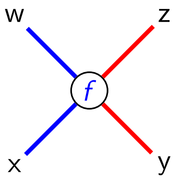

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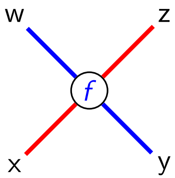

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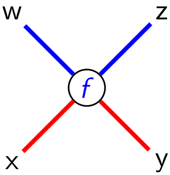

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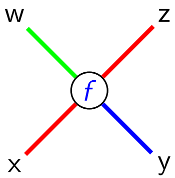

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- Third reduction: Via a gadget construction.

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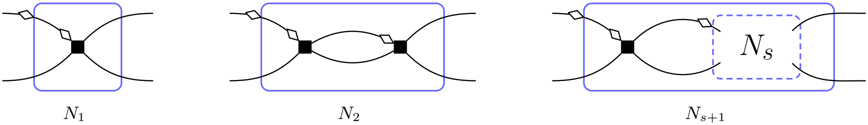
Polynomial Interpolation Step: Recursive Construction

$$\text{Holant}_3(G; \langle 2, 1, 0, 1, 0\rangle) \leq_T \text{Holant}_3(G_s; \langle 0, 1, 1, 0, 0\rangle)$$



Vertices are assigned $\langle 0, 1, 1, 0, 0\rangle$.

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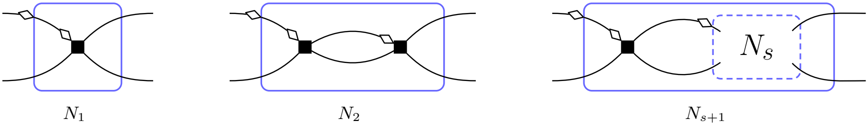

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Let f_s be the function corresponding to N_s. Then $f_s = M^s f_0$, where

$$M = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad f_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Obviously $f_1 = \langle 0,1,1,0,0\rangle$.

Polynomial Interpolation Step: Eigenvectors and Eigenvalues

Spectral decomposition $M = P\Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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Let $x = 2^{2s}$. Then

$$f_{2s} = P\Lambda^{2s}P^{-1}f_0 = P \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}f_0 = \begin{bmatrix} \frac{x-1}{3} + 1 \\ \frac{x-1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

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Note $f(4) = \langle 2, 1, 0, 1, 0 \rangle$.

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(Side note: picking $s = 1$ so that $x = 4$ only works when $\kappa = 3$.)

Polynomial Interpolation Step: The Interpolation

$$\text{Holant}_3(-; \langle 2,1,0,1,0 \rangle) \leq_T \text{Holant}_3(-; \langle 0,1,1,0,0 \rangle)$$

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If G has n vertices, then

$$p(G, x) = \text{Holant}_3(G; f(x)) \in \mathbb{Z}[x]$$

has degree n.

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Let G_{2s} be the graph obtained by replacing every vertex in G with N_{2s}. Then $\text{Holant}_3(G_{2s}; \langle 0,1,1,0,0\rangle) = p(G, 2^{2s})$.

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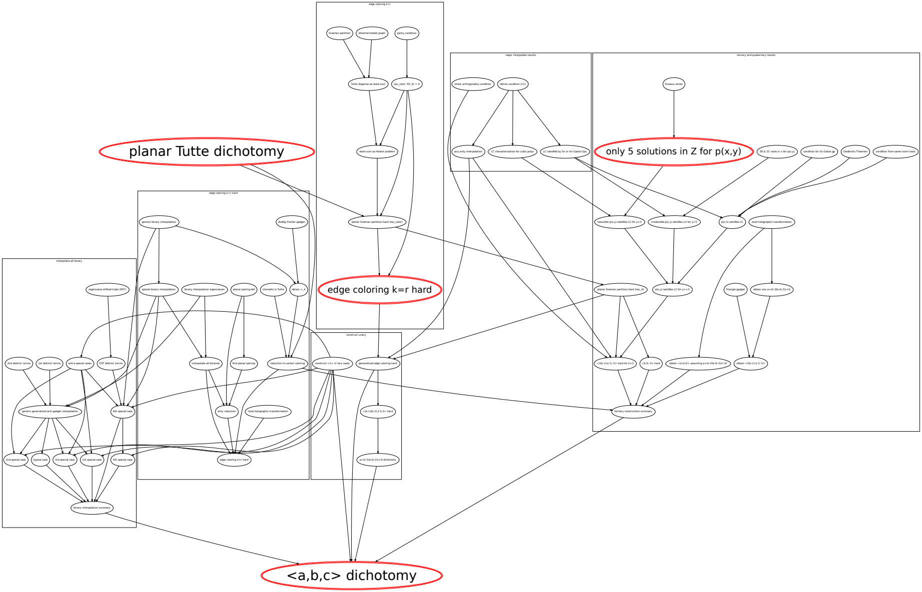
Using oracle for $\text{Holant}_3(-; \langle 0,1,1,0,0\rangle)$, evaluate $p(G, x)$ at $n+1$ distinct points $x = 2^{2s}$ for $0 \leq s \leq n$.

By polynomial interpolation, efficiently compute the coefficients of $p(G, x)$. QED.

Dichotomy of $\text{Holant}_\kappa(-; \langle a, b, c\rangle)$



Interpolating Univariate Polynomials

Let $p_d(X) = c_0 + c_1 X + c_2 X^2 + \cdots + c_{d-1} X^{d-1} + c_d X^d \in \mathbb{Z}[X]$.

$$\text{Can interpolate } p_d(X) \text{ from}$$
$$p_d(x_0), p_d(x_1), p_d(x_2), \ldots, p_d(x_{d-1}), p_d(x_d)$$
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$$x_0, x_1, x_2, \ldots, x_{d-1}, x_d \text{ are distinct}$$

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x is not a root of unity

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Interpolating Multivariate Polynomials

Let

$$q_d(X, Y, Z) = c_{0,0,d}\ X^0 Y^0 Z^d + c_{0,1,d-1}\ X^0 Y^1 Z^{d-1} + \cdots + c_{d,0,0}\ X^d Y^0 Z^0$$

in $\mathbb{Z}[X, Y, Z]$ be a homogeneous polynomial of degree d.

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lattice condition

Lattice Condition

Definition

We say that $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in \mathbb{C} - \{0\}$ satisfy the lattice condition if

$$\forall x \in \mathbb{Z}^\ell - \{\mathbf{0}\} \qquad \text{with} \qquad \sum_{i=1}^{\ell} x_i = 0,$$

we have

$$\prod_{i=1}^{\ell} \lambda_i^{x_i} \neq 1.$$

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Example

For any $i, j, k \in \mathbb{Z}$ such that

- $i + j + k = 0$ and
- $(i, j, k) \neq (0, 0, 0)$,

it follows that

$$2^i 3^j 5^k \neq 1.$$

Actual "Lattice Condition" Example

Want to prove:

For all integers $y \geq 4$, the roots of

$$p(x, y) = x^5 - (2y+1)x^3 - (y^2+2)x^2 + (y-1)yx + y^3.$$

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satisfy the lattice condition.

Lemma

Let $p(x) \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 2$. If

1. *the Galois group of p over $\mathbb{Q}$ is S_n or A_n and*
2. *the roots of p do not all have the same complex norm,*

then the roots of p satisfy the lattice condition.

Galois group of p over $\mathbb{Q}$ is S_n or A_n

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$$p \text{ is irreducible over } \mathbb{Q}$$
$$\Updownarrow \text{ (Gauss' Lemma)}$$
$$p \text{ is irreducible over } \mathbb{Z}$$
$$\Downarrow$$
$$p \text{ has no root in } \mathbb{Z}$$

Factorizations and Roots

Galois group of p over $\mathbb{Q}$ is S_n or A_n

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p is irreducible over $\mathbb{Q}$

$$\Updownarrow \text{(Gauss' Lemma)}$$

p is irreducible over $\mathbb{Z}$

$$\Downarrow$$

p has no root in $\mathbb{Z}$

What are the known nontrivial factorizations of $p(x, y)$?
What are the known integer roots of $p(x, y)$?

$$p(x,y) = \begin{cases} (x-1)(x^4+x^3+2x^2-x+1) & y=-1 \\ x^2(x^3-x-2) & y=0 \\ (x+1)(x^4-x^3-2x^2-x+1) & y=1 \\ (x-1)(x^2-x-4)(x^2+2x+2) & y=2 \\ (x-3)(x^4+3x^3+2x^2-5x-9) & y=3. \end{cases}$$

Theorem (Siegel's Theorem)

Any smooth algebraic curve of genus $g > 0$ defined by a polynomial in $\mathbb{Z}[x, y]$ has only finitely many integer solutions.

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Any smooth algebraic curve of genus $g > 0$ defined by a polynomial in $\mathbb{Z}[x, y]$ has only finitely many integer solutions.

- p(x,y) has genus 3, satisfies the condition
- Bad news is that Siegel's theorem is not effective
- Several effective versions, but the best bound we found is 10^{20000}
- Integer solutions could be enormous

Pell's Equation (genus 0)

$$x^2 - 991y^2 = 1$$

Diophantine Equations with Enormous Solutions

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Smallest solution:

$$(379516400906811930638014896080,$$
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Diophantine Equations with Enormous Solutions

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Smallest solution:

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Next smallest solution:

$$(288065397114519999215772221121510725946342952839946398732799,$$
$$9150698914859994783783151874415159820056535806397752666720)$$

What We Believe vs. What We Can Prove

Conjecture

For any integer $y \geq 4$, $p(x, y)$ *is irreducible in* $\mathbb{Z}[x]$.

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Don't know how to prove this.

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For any integer $y \geq 4$, $p(x, y)$ *is irreducible in* $\mathbb{Z}[x]$.

Don't know how to prove this.

Lemma

Only integer solutions to $p(x, y) = 0$ *are*

$$(1, -1), (0, 0), (-1, 1), (1, 2), (3, 3).$$

Puiseux Series

Puiseux series expansions for $p(x, y)$ are

$$y_1(x) = x^2 + 2x^{-1} + 2x^{-2} - 6x^{-4} - 18x^{-5} + O(x^{-6}),$$

$$y_2(x) = \quad x^{3/2} - \frac{1}{2}x + \frac{1}{8}x^{1/2} - \frac{65}{128}x^{-1/2} - x^{-1} - \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2}),$$

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We pick functions $g_i(x,y)$ such that

- (a,b) integer solution to $p(x,y) = 0$ implies $g_i(a,b) \in \mathbb{Z}$
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Consider $g_2(x, y) = y^2 + xy - x^3 + x$

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If $|a| > 16$, then $g_2(a, y_2(a))$ is not an integer.

Thank You

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Paper and slides available on my website:
`www.cs.wisc.edu/~tdw`