First Order Logic
Part 1

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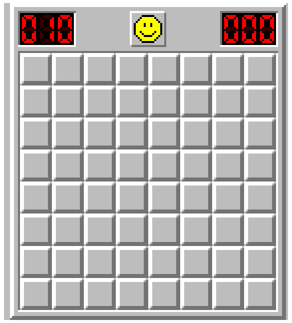
[Based on slides from Burr Settles and Jerry Zhu]

Problems with propositional logic

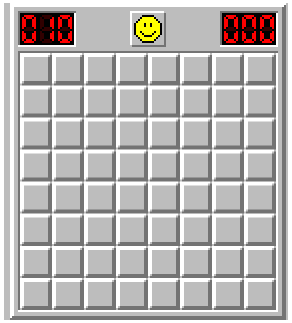
- Consider the game “minesweeper” on a 10x10 field with only one landmine.



- How do you express the knowledge, with propositional logic, that the squares adjacent to the landmine will display the number 1?

Problems with propositional logic

- Consider the game "minesweeper" on a 10x10 field with only one landmine.



- How do you express the knowledge, with propositional logic, that the squares adjacent to the landmine will display the number 1?
- Intuitively with a rule like

$$landmine(x,y) \Rightarrow number1(neighbors(x,y))$$

but propositional logic cannot do this…

Problems with propositional logic

- Propositional logic has to say, e.g. for cell (3,4):
  - Landmine_3_4 $\Rightarrow$ number1_2_3
  - Landmine_3_4 $\Rightarrow$ number1_2_4
  - Landmine_3_4 $\Rightarrow$ number1_2_5
  - Landmine_3_4 $\Rightarrow$ number1_3_3
  - Landmine_3_4 $\Rightarrow$ number1_3_5
  - Landmine_3_4 $\Rightarrow$ number1_4_3
  - Landmine_3_4 $\Rightarrow$ number1_4_4
  - Landmine_3_4 $\Rightarrow$ number1_4_5
  - And similarly for each of Landmine_1_1, Landmine_1_2, Landmine_1_3, …, Landmine_10_10!
- Difficult to express large domains concisely
- Don’t have objects and relations
- First Order Logic is a powerful upgrade

Ontological commitment

- Logics are characterized by what they consider to be 'primitives'

Logic	Primitives	Available Knowledge
Propositional	facts	true/false/unknown
First-Order	facts, objects, relations	true/false/unknown
Temporal	facts, objects, relations, times	true/false/unknown
Probability Theory	facts	degree of belief 0…1
Fuzzy	degree of truth	degree of belief 0…1

First Order Logic syntax

- **Term**: an object in the world
  - **Constant**: Jerry, 2, Madison, Green, …
  - **Variables**: x, y, a, b, c, …
  - **Function**($term_1$, …, $term_n$)
    - Sqrt(9), Distance(Madison, Chicago)
    - Maps one or more objects to another object
    - Can refer to an unnamed object: LeftLeg(John)
    - Represents a user defined functional relation

- A **ground term** is a term without variables.

FOL syntax

- **Atom**: smallest T/F expression
  - **Predicate**($term_1$, …, $term_n$)
    - Teacher(Jerry, you), Bigger(sqrt(2), x)
    - Convention: read “Jerry (is)Teacher(of) you”
    - Maps one or more objects to a truth value
    - Represents a user defined relation
  - **$term_1$ = $term_2$**
    - Radius(Earth)=6400km, 1=2
    - Represents the equality relation when two terms refer to the same object

FOL syntax

- **Sentence**: T/F expression
  - Atom
  - Complex sentence using connectives: $\wedge \vee \neg \Rightarrow \Leftrightarrow$
    - Spouse(Jerry, Jing) $\Rightarrow$ Spouse(Jing, Jerry)
    - Less(11,22) $\wedge$ Less(22,33)
  - Complex sentence using quantifiers $\forall$, $\exists$
- Sentences are evaluated under an interpretation
  - Which objects are referred to by constant symbols
  - Which objects are referred to by function symbols
  - What subsets defines the predicates

FOL quantifiers

- Universal quantifier: $\forall$
- Sentence is true **for all** values of x in the domain of variable x.

- Main connective typically is $\Rightarrow$
  - Forms if-then rules
  - "all humans are mammals"

    $\forall$**x human(x)** $\Rightarrow$ **mammal(x)**

  - Means if x is a human, then x is a mammal

FOL quantifiers

$$\forall x\ \mathbf{human(x)} \Rightarrow \mathbf{mammal(x)}$$

- It's a big AND: Equivalent to the conjunction of all the instantiations of variable x:

$$(\mathbf{human(Jerry)} \Rightarrow \mathbf{mammal(Jerry)}) \wedge$$
$$(\mathbf{human(Jing)} \Rightarrow \mathbf{mammal(Jing)}) \wedge$$
$$(\mathbf{human(laptop)} \Rightarrow \mathbf{mammal(laptop)}) \wedge \ldots$$

- Common mistake is to use $\wedge$ as main connective

$$\forall x\ \mathbf{human(x)} \wedge \mathbf{mammal(x)}$$

- This means everything is human and a mammal!

$$(\mathbf{human(Jerry)} \wedge \mathbf{mammal(Jerry)}) \wedge$$
$$(\mathbf{human(Jing)} \wedge \mathbf{mammal(Jing)}) \wedge$$
$$(\mathbf{human(laptop)} \wedge \mathbf{mammal(laptop)}) \wedge \ldots$$

FOL quantifiers

- Existential quantifier: ∃
- Sentence is true **for some** value of x in the domain of variable x.

- Main connective typically is ∧
  - "some humans are male"

    $$\exists x\ \mathbf{human(x)} \wedge \mathbf{male(x)}$$

  - Means there is an x who is a human and is a male

FOL quantifiers

$$\exists x \; \texttt{human(x)} \wedge \texttt{male(x)}$$

- It's a big OR: Equivalent to the disjunction of all the instantiations of variable x:

$$(\texttt{human(Jerry)} \wedge \texttt{male(Jerry)}) \vee$$
$$(\texttt{human(Jing)} \wedge \texttt{male(Jing)}) \vee$$
$$(\texttt{human(laptop)} \wedge \texttt{male(laptop)}) \vee \ldots$$

- Common mistake is to use $\Rightarrow$ as main connective
  - "Some pig can fly"

$$\exists x \; \texttt{pig(x)} \Rightarrow \texttt{fly(x)} \quad \textbf{(wrong)}$$

FOL quantifiers

$$\exists x\ \mathtt{human(x)} \land \mathtt{male(x)}$$

- It's a big OR: Equivalent to the disjunction of all the instantiations of variable x:

$$(\mathtt{human(Jerry)} \land \mathtt{male(Jerry)}) \lor$$
$$(\mathtt{human(Jing)} \land \mathtt{male(Jing)}) \lor$$
$$(\mathtt{human(laptop)} \land \mathtt{male(laptop)}) \lor \ldots$$

- Common mistake is to use $\Rightarrow$ as main connective
  - "Some pig can fly"

$$\exists x\ \mathtt{pig(x)} \Rightarrow \mathtt{fly(x)}\ \textbf{(wrong)}$$

- This is true if there is something not a pig!

$$(\mathtt{pig(Jerry)} \Rightarrow \mathtt{fly(Jerry)}) \lor$$
$$(\mathtt{pig(laptop)} \Rightarrow \mathtt{fly(laptop)}) \lor \ldots$$

FOL quantifiers

- Properties of quantifiers:
  - $\forall$**x** $\forall$**y** is the same as $\forall$**y** $\forall$**x**
  - $\exists$**x** $\exists$**y** is the same as $\exists$**y** $\exists$**x**

- Example:
  - $\forall$**x** $\forall$**y** **likes(x,y)**

    Everyone likes everyone.
  - $\forall$**y** $\forall$**x** **likes(x,y)**

    Everyone is liked by everyone.

FOL quantifiers

- Properties of quantifiers:
  - $\forall$**x** $\exists$**y** is not the same as $\exists$**y** $\forall$**x**
  - $\exists$**x** $\forall$**y** is not the same as $\forall$**y** $\exists$**x**

- Example:
  - $\forall$**x** $\exists$**y** **`likes(x,y)`**

    Everyone likes someone (can be different).
  - $\exists$**y** $\forall$**x** **`likes(x,y)`**

    There is someone who is liked by everyone.

FOL quantifiers

- Properties of quantifiers:
  - $\forall$**`x P(x)`** when negated becomes $\exists$**`x`** $\neg$**`P(x)`**
  - $\exists$**`x P(x)`** when negated becomes $\forall$**`x`** $\neg$**`P(x)`**

- Example:
  - $\forall$**`x sleep(x)`**

    Everybody sleeps.
  - $\exists$**`x`** $\neg$**`sleep(x)`**

    Somebody does not sleep.

FOL quantifiers

- Properties of quantifiers:
  - $\forall$**x P(x)** is the same as $\neg\exists$**x** $\neg$**P(x)**
  - $\exists$**x P(x)** is the same as $\neg\forall$**x** $\neg$**P(x)**

- Example:
  - $\forall$**x sleep(x)**

    Everybody sleeps.
  - $\neg\exists$**x** $\neg$**sleep(x)**

    There does not exist someone who does not sleep.

FOL syntax

- A free variable is a variable that is not bound by an quantifier, e.g. `∃y Likes(x,y)`: `x` is free, `y` is bound
- A well-formed formula (wff) is a sentence in which all variables are quantified (no free variable)

- Short summary so far:
  - **Constants:** Bob, 2, Madison, …
  - **Variables:** *x, y, a, b, c*, …
  - **Functions:** Income, Address, Sqrt, …
  - **Predicates:** Teacher, Sisters, Even, Prime…
  - **Connectives:** $\wedge \vee \neg \Rightarrow \Leftrightarrow$
  - **Equality:** $=$
  - **Quantifiers:** $\forall \exists$

More summary

- **Term**: constant, variable, function. Denotes an object. (A ground term has no variables)
- **Atom**: the smallest expression assigned a truth value. Predicate and =
- **Sentence**: an atom, sentence with connectives, sentence with quantifiers. Assigned a truth value
- **Well-formed formula** (wff): a sentence in which all variables are quantified