Bayesian Networks Part 1

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Computer Sciences 760
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Some of the slides in these lectures have been adapted/borrowed from materials developed by Mark Craven, David Page, Jude Shavlik, Tom Mitchell, Nina Balcan, Matt Gormley, Elad Hazan, Tom Dietterich, and Pedro Domingos.

Goals for the lecture

you should understand the following concepts

- the Bayesian network representation
- inference by enumeration
- the parameter learning task for Bayes nets
- the structure learning task for Bayes nets
- maximum likelihood estimation
- Laplace estimates
- *m*-estimates

Bayesian network example

- Consider the following 5 binary random variables:

  B = a burglary occurs at your house

  E = an earthquake occurs at your house

  A = the alarm goes off

  J = John calls to report the alarm

  M = Mary calls to report the alarm

- Suppose we want to answer queries like what is $P(B \mid M, J)$?

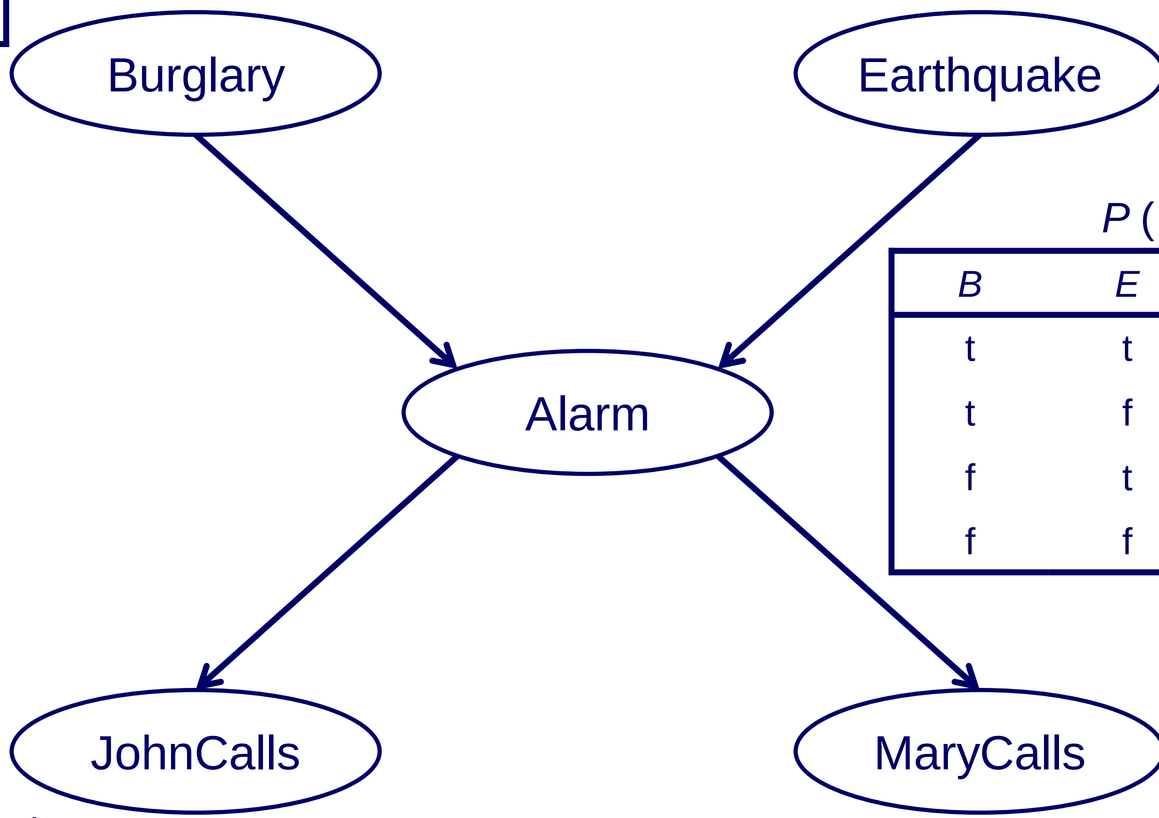
Bayesian network example

P (*B*)

t	f
0.001	0.999

P (*E*)

t	f
0.001	0.999



P (*A* | *B*, *E*)

B	*E*	t	f
t	t	0.95	0.05
t	f	0.94	0.06
f	t	0.29	0.71
f	f	0.001	0.999

P (*J* | *A*)

A	t	f
t	0.9	0.1
f	0.05	0.95

P (*M* | *A*)

A	t	f
t	0.7	0.3
f	0.01	0.99

Bayesian networks

- a BN consists of a Directed Acyclic Graph (DAG) and a set of conditional probability distributions

- in the DAG
  - each node denotes random a variable
  - each edge from X to Y represents that X *directly influences* Y
  - formally: each variable X is independent of its non-descendants given its parents

- each node X has a *conditional probability distribution* (CPD) representing $P(X \mid Parents(X))$

Bayesian networks

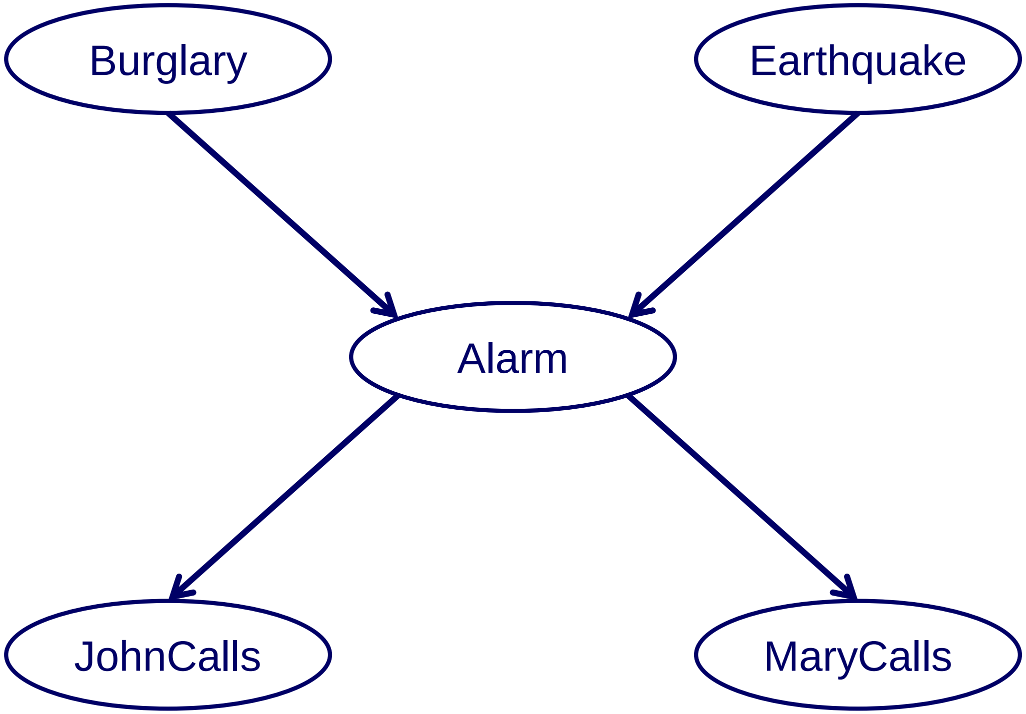
- using the chain rule, a joint probability distribution can be expressed as

$$P(X_1,\ldots,X_n) = P(X_1)\prod_{i=2}^{n} P(X_i \mid X_1,\ldots,X_{i-1})$$

- a BN provides a compact representation of a joint probability distribution

$$P(X_1,\ldots,X_n) = P(X_1)\prod_{i=2}^{n} P(X_i \mid Parents(X_i))$$

Bayesian networks



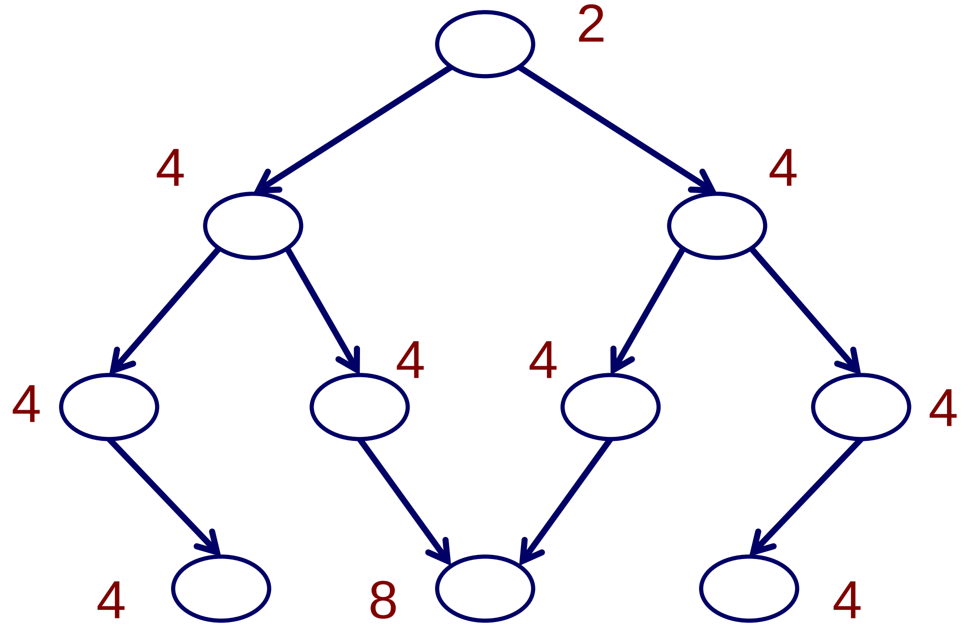
$$
\begin{aligned}
&P(B, E, A, J, M) \\
&= P(B) \\
&\times P(E) \\
&\times P(A \mid B, E) \\
&\times P(J \mid A) \\
&\times P(M \mid A)
\end{aligned}
$$

- a standard representation of the joint distribution for the Alarm example has $2^5 = 32$ parameters
- the BN representation of this distribution has 20 parameters

Bayesian networks

- consider a case with 10 binary random variables

- How many parameters does a BN with the following graph structure have?



= 42

- How many parameters does the standard table representation of the joint distribution have?

= 1024

Advantages of the Bayesian network representation

- Captures independence and conditional independence where they exist
- Encodes the relevant portion of the full joint among variables where dependencies exist
- Uses a graphical representation which lends insight into the complexity of inference

The inference task in Bayesian networks

Given: values for some variables in the network (*evidence*), and a set of *query* variables

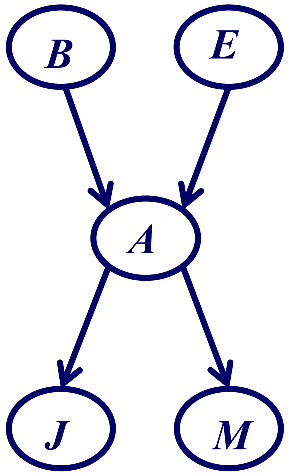
Do: compute the posterior distribution over the query variables

- variables that are neither evidence variables nor query variables are *hidden* variables
- the BN representation is flexible enough that any set can be the evidence variables and any set can be the query variables

Inference by enumeration

- let a denote A=true, and $\neg a$ denote A=false
- suppose we're given the query: $P(b \mid j, m)$

  "probability the house is being burglarized given that John and Mary both called"
- from the graph structure we can first compute:



$$P(b, j, m) = \sum_{e, \neg e} \sum_{a, \neg a} P(b) P(E) P(A \mid b, E) P(j \mid A) P(m \mid A)$$

sum over possible values for E and A variables ($e, \neg e, a, \neg a$)

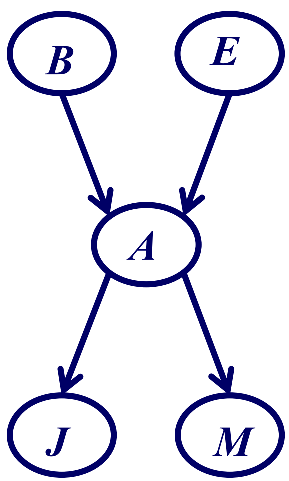
Inference by enumeration

$$P(b, j, m) = \sum_{e,\neg e}\sum_{a,\neg a} P(b)P(E)P(A \mid b, E)P(j \mid A)P(m \mid A)$$

$$= P(b)\sum_{e,\neg e}\sum_{a,\neg a} P(E)P(A \mid b, E)P(j \mid A)P(m \mid A)$$

P(B)
0.001

P(E)
0.001



B	E	P(A)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J)
t	0.9
f	0.05

A	P(M)
t	0.7
f	0.01

$B \quad E \quad A \quad J \quad M$

$$\begin{aligned} = 0.001 \times (&0.001 \times 0.95 \times 0.9 \times 0.7 + && e,\ a \\ &0.001 \times 0.05 \times 0.05 \times 0.01 + && e,\ \neg a \\ &0.999 \times 0.94 \times 0.9 \times 0.7 + && \neg e,\ a \\ &0.999 \times 0.06 \times 0.05 \times 0.01) && \neg e,\ \neg a \end{aligned}$$

Inference by enumeration

- now do equivalent calculation for $P(\neg b,\ j,\ m)$
- and determine $P(b \mid j,\ m)$

$$P(b \mid j,m) = \frac{P(b,j,m)}{P(j,m)} = \frac{P(b,j,m)}{P(b,j,m) + P(\neg b,j,m)}$$

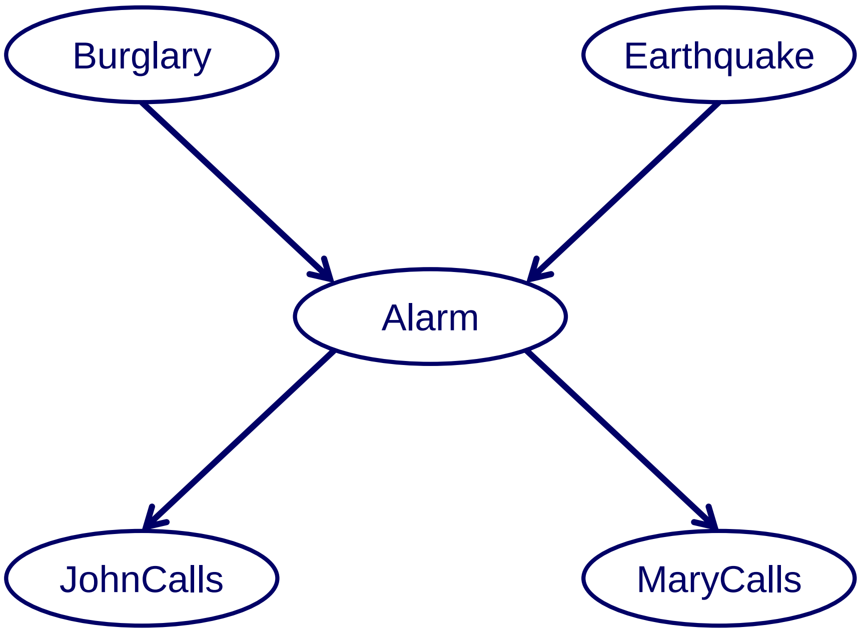
Comments on BN inference

- *inference by enumeration* is an *exact* method (i.e. it computes the exact answer to a given query)
- it requires summing over a joint distribution whose size is exponential in the number of variables
- in many cases we can do exact inference efficiently in large networks
  - key insight: save computation by pushing sums inward
- in general, the Bayes net inference problem is NP-hard
- there are also methods for approximate inference – these get an answer which is "close"
- in general, the approximate inference problem is NP-hard also, but approximate methods work well for many real-world problems

The parameter learning task

- Given: a set of training instances, the graph structure of a BN

B	E	A	J	M
f	f	f	t	f
f	t	f	f	f
f	f	t	f	t
		…		



- Do: infer the parameters of the CPDs

The structure learning task

- Given: a set of training instances

B	E	A	J	M
f	f	f	t	f
f	t	f	f	f
f	f	t	f	t
		…		

- Do: infer the graph structure (and perhaps the parameters of the CPDs too)

Parameter learning and maximum likelihood estimation

- *maximum likelihood estimation* (MLE)
  - given a model structure (e.g. a Bayes net graph) G and a set of data D
  - set the model parameters θ to maximize $P(D \mid G, \theta)$

- i.e. make the data D look <u>as likely as possible</u> under the model $P(D \mid G, \theta)$

Maximum likelihood estimation

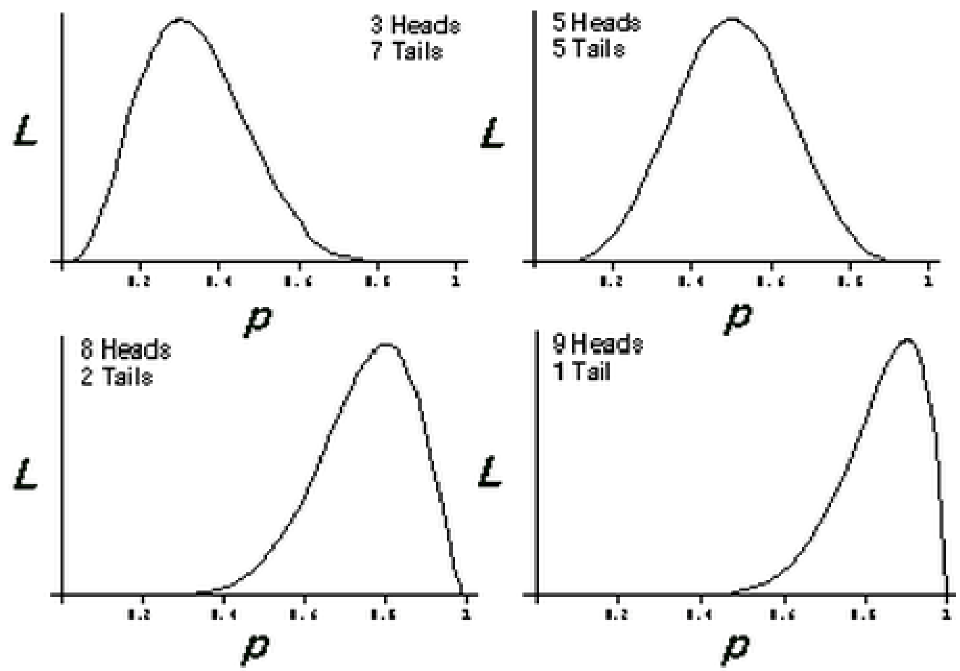
consider trying to estimate the parameter θ (probability of heads) of a biased coin from a sequence of flips

$$\boldsymbol{x} = \{1,1,1,0,1,0,0,1,0,1\}$$

the likelihood function for θ is given by:

$$L(\theta : x_1, \ldots, x_n) = \theta^{x_1}(1-\theta)^{1-x_1} \cdots \theta^{x_n}(1-\theta)^{1-x_n}$$
$$= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

for h heads in n flips
the MLE is h/n



MLE in a Bayes net

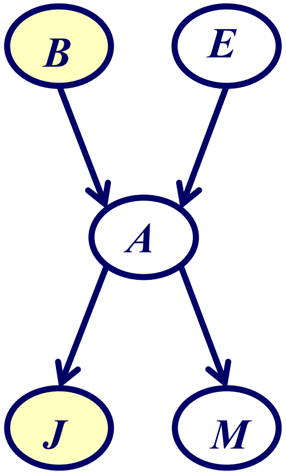
$$L(\theta : D, G) = P(D \mid G, \theta) = \prod_{d \in D} P(x_1^{(d)}, x_2^{(d)}, \ldots, x_n^{(d)})$$

$$= \prod_{d \in D} \prod_{i} P(x_i^{(d)} \mid Parents(x_i^{(d)}))$$

$$= \prod_{i} \underbrace{\left(\prod_{d \in D} P(x_i^{(d)} \mid Parents(x_i^{(d)})) \right)}_{\text{independent parameter learning problem for each CPD}}$$

Maximum likelihood estimation

now consider estimating the CPD parameters for B and J in the alarm network given the following data set



B	E	A	J	M
f	f	f	t	f
f	t	f	f	f
f	f	f	t	t
t	f	f	f	t
f	f	t	t	f
f	f	t	f	t
f	f	t	t	t
f	f	t	t	t

$$P(b) = \frac{1}{8} = 0.125$$

$$P(\neg b) = \frac{7}{8} = 0.875$$

$$P(j \mid a) = \frac{3}{4} = 0.75$$

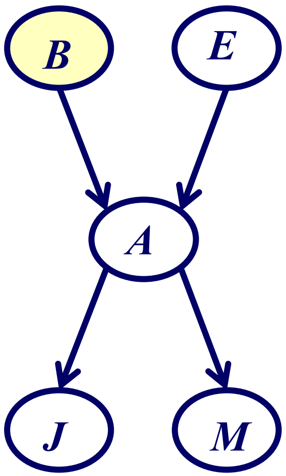
$$P(\neg j \mid a) = \frac{1}{4} = 0.25$$

$$P(j \mid \neg a) = \frac{2}{4} = 0.5$$

$$P(\neg j \mid \neg a) = \frac{2}{4} = 0.5$$

Maximum likelihood estimation

suppose instead, our data set was this…



B	E	A	J	M
f	f	f	t	f
f	t	f	f	f
f	f	f	t	t
f	f	f	f	t
f	f	t	t	f
f	f	t	f	t
f	f	t	t	t
f	f	t	t	t

$$\boxed{P(b) = \frac{0}{8} = 0}$$

$$P(\neg b) = \frac{8}{8} = 1$$

do we really want to set this to 0?

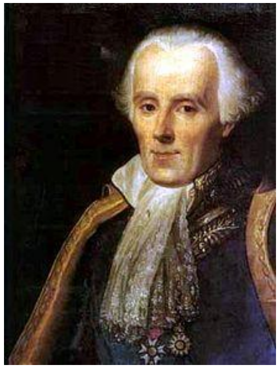
Maximum a posteriori (MAP) estimation

- instead of estimating parameters strictly from the data, we could start with some prior belief for each

- for example, we could use *Laplace estimates*

$$P(X = x) = \frac{n_x + 1}{\sum_{v \in \text{Values}(X)} (n_v + 1)}$$

pseudocounts



- where n_v represents the number of occurrences of value v

Maximum a posteriori estimation

a more general form: *m-estimates*

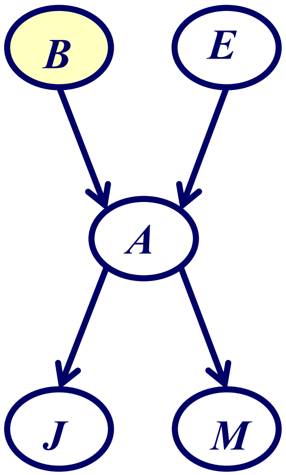
$$P(X = x) = \frac{n_x + p_x m}{\left(\sum_{v \in \text{Values}(X)} n_v \right) + m}$$

prior probability of value x

number of "virtual" instances

M-estimates example

now let's estimate parameters for B using $m=4$ and $p_b=0.25$



B	E	A	J	M
f	f	f	t	f
f	t	f	f	f
f	f	f	t	t
f	f	f	f	t
f	f	t	t	f
f	f	t	f	t
f	f	t	t	t
f	f	t	t	t

$$P(b) = \frac{0 + 0.25 \times 4}{8 + 4} = \frac{1}{12} = 0.08 \qquad P(\neg b) = \frac{8 + 0.75 \times 4}{8 + 4} = \frac{11}{12} = 0.92$$