Problem 1. These look like true/false questions, but they are really short answer questions. Decide if the following statements are TRUE or FALSE and give short reasons for your choice. (10 points)

a) \( \{0^n1^n \mid n \geq 0\} \cap R \), where \( R \) is a regular language, is never regular.

b) Suppose \( \Sigma = \{0,1\} \), and let \( \text{sort}(x) \) be the function that reorders the symbols in \( x \) in numerical order. Let \( \text{sort}(L) = \{\text{sort}(x) \mid x \in L\} \). For example, if \( L=\{0,1,01,10,101\} \), then \( \text{sort}(L)=\{0,1,01,11,001\} \). Regular languages are closed under \( \text{sort} \).

Problem 2. These look like true/false questions, but they are really short answer questions. Decide if the following statements are TRUE or FALSE and give short reasons for your choice. (10 points)

a) If \( L \) is a finite context-free language, then \( \bar{L} \) (the complement of \( L \)) must be context-free.

b) If every state of an NFA \( N \) is accepting, then \( L(N) = \Sigma^* \).

Problem 3. Design a DFA that accepts strings over \( \{0,1\} \) containing 101 as a substring. (10 points)

Note: Draw the graph. Do not give the transition table.

Problem 4. Prove that there exists an integer whose decimal representation consists entirely of 1’s, and which is divisible by 1987. (10 points)

Hint: If \( p \) is a prime such that \( p|x(y) \) and \( p \) does not divide \( y \), then \( p|x \). You should only consider the remainders modulo 1987 of integers whose decimal representation consists entirely of 1’s and use the pigeonhole principle.

Problem 5. Show that regular languages are not closed under infinite union. (10 points)

Note: Infinite union means a union of an infinite family of sets. For this problem, you need to come up with an infinite family of regular sets whose union is not regular.

Hint: Start with a non-regular language and break it down into an infinite family of regular sets.

Problem 6. Design a PDA for strings over \( \{0,1\} \) of the form \( 0^n1^{2n} \), where \( n \geq 0 \). (10 points)

Note: You can either give a PDA that accepts by final state or by empty stack.
Problem 7. Design a PDA for the set of all palindromes over \{0,1\}. (10 points)

Note: You can either give a PDA that accepts by final state or by empty stack.
Hint: You should first write out the corresponding CFG and then convert it to a PDA.

Problem 8. Prove that \(2^{243} + 1\) is divisible by 729. (10 points)

Hint: Try to prove the more general statement that \(2^{3^n} + 1\) is divisible by \(3^{n+1}\). Recall that \(a^3 + b^3 = (a + b)(a^2 - ab + b^2)\) and \(a^3 = (a^i)^3\).

Problem 9. Minimize the DFA shown in Figure 1 by marking distinguishable states in a table and then draw the minimized DFA. (10 points)

Problem 10. Prove that the set of all strings over \{0,1\} of the form \(w\bar{w}\), where \(\bar{w}\) is formed from \(w\) by replacing all 0’s by 1’s, and vice-versa is not regular. For example, \(\bar{100} = 100\) and \(01100\) is an example of a string in the language. (10 points)