The Pumping Lemma and Closure Properties

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Tentative Schedule

- HW #1: Out (07/03), Due (07/11)
- HW #2: Out (07/10), Due (07/18)
- HW #3: Out (07/17), Due (07/25)
- Midterm: 07/31 (in class)
- HW #4: Out (07/31), Due (08/08)
- Tentative grades out by 08/12.
- Final: 08/18 (?)

The Infiniteness Problem

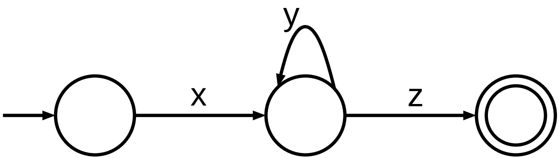
Question

Is a given regular language L infinite?

- Start with a DFA for the language.
- **Key idea:** If the DFA has n states, and the language contains any string of length n or more, then the language is infinite.
- Otherwise, the language is surely finite.
  - Limited to strings of length n or less.

Proof of Key Idea

- If an n-state DFA accepts a string w of length n or more, then there must be a state that appears twice on the path labeled w from the start state to a final state.
    - **Note:** Pigeonhole principle! $\ddot\smile$
- Because there are at least $n + 1$ states along the path.



- Since y is not ε, we see an infinite number of strings in L of the form xy^iz for all $i \geq 0$.

The Pumping Lemma

Theorem

For every regular language L, there is an integer n such that for every string $w \in L$ of length $\geq n$, we can write $\mathrm{w} = \mathrm{xyz}$ such that:

- $|xy| \leq n$.
- $|y| > 0$.
- For all $i \geq 0$, xy^iz is in L.

The Pumping Lemma: Examples

Question

Prove that the language $L = \{0^k 1^k \mid k \geq 1\}$ is not regular.

- Proof by contradiction. Suppose it were, and let a DFA with n states accept all strings in L.
- Choose the string $w = 0^n 1^n$. We can write $w = xyz$ where x and y consist of 0's, and $y \neq \varepsilon$.
- But then xyyz would be in L, and this string has more 0's than 1's!
- Choice of the proper string is very important!
  - **Example:** $w = (01)^n$ does not work!
- Same argument as above also works for $L = \{w \mid w$ has equal number of 0's and 1's.$\}$

The Pumping Lemma: Examples

Question

Prove that the language $L = \{ww \mid w \in \{0,1\}^*\}$ is not regular.

The Pumping Lemma: Examples

Question

Prove that the language $L = \{ww \mid w \in \{0,1\}^*\}$ is not regular.

- Suppose it were, and let a DFA with n states accept all strings in L.
- Choose the string $w = 0^n10^n1$. We can write $w = xyz$ where x and y consist of 0's, and $y \neq \varepsilon$.
- But then xyyz would be in L!
- Note that the string $w = 0^n0^n$ does not work.

The Pumping Lemma: Examples

Question

Prove that the language $L = \{1^{k^2} \mid k \geq 0\}$ is not regular.

The Pumping Lemma: Examples

Question

Prove that the language $L = \{1^{k^2} \mid k \geq 0\}$ is not regular.

- Suppose it were, and let a DFA with n states accept all strings in L.
- Choose the string $w = 1^{n^2}$. We can write $w = xyz$. Consider the string $s = xyyz$.
- We know that $|xy| \leq n$ and thus $|y| \leq n$. So no. of 1's in xyyz is $n^2+n < n^2+2n+1$!
- The parameter n is often called the pumping length.

The Pumping Lemma: Examples

Question

Prove that the language $L = \{0^i 1^j \mid i > j\}$ is not regular.

The Pumping Lemma: Examples

- Sometimes pumping down is useful as well!

Question

Prove that the language $L = \{0^i 1^j \mid i > j\}$ is not regular.

- Let n be the pumping length, i.e., suppose there exists a DFA with n states that accepts all strings in L.
- Choose the string $w = 0^{n+1}1^n$. We can write $w = xyz$ where x and y consist of 0's, and $y \neq \varepsilon$.
- The pumping lemma states that all strings $xy^iz \in L$, even for $i = 0$!
- The string $w = xz$ cannot have more 0's than 1's.

The Pumping Lemma: Examples

Question

Prove that the language $L = \{1^p \mid \text{where } p \text{ is prime}\}$ is not regular.

The Pumping Lemma: Examples

Question

Prove that the language $L = \{1^p \mid \text{where } p \text{ is prime}\}$ is not regular.

- Consider some primt $q \geq n + 2$, where n is the pumping length.
- Choose the string $w = 1^q$. We can write $w = xyz$ such that $y \neq \varepsilon$ and $|xy| \leq n$.
- Let $|y| = m$. Then $|xz| = q - m$. Consider the string $s = xy^{q-m}z$ which is in L by the pumping lemma.
- $|xy^{q-m}z| = |xz| + (q-m)|y| = q-m + (q-m)m = (m+1)(q-m)$.
- Note that $m+1 > 1$, as $y \neq \varepsilon$.
- Also note that $q \geq n + 2$, and so $q - m > 1$.

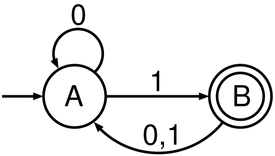
Closure Properties

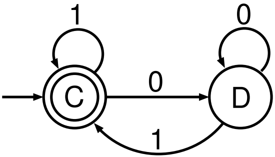
- A closure property of a language class says that given languages in the class, an operator (e.g., union) produces another language in the same class.
- **Example:** We saw that regular languages are closed under union, concatenation and Kleene closure (star) operations.
- We will see more examples: intersection, difference, reversal, homomorphism, inverse homomorphism.

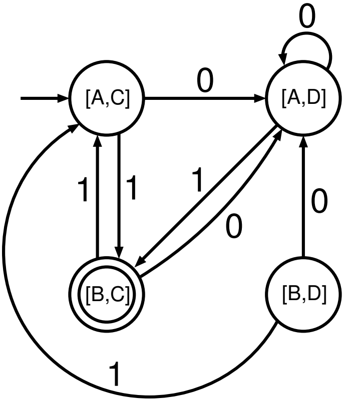
Closure Properties: Intersection

- Construct the product DFA from DFA's for L and M.
- Let these DFA's have sets of states Q and R respectively.
- Product DFA has set of states $Q \times R$.
  - i.e., pairs [q,r] with $q \in Q$ and $r \in Q$.
- Start state = $[q_0, r_0]$ (the start states of the DFA for L and M).
- **Transitions:** $\delta([q,r],a) = [\delta_L(q,a),\delta_M(r,a)]$.
  - δ_L, δ_M are the transition functions for the DFA's of L and M.
  - i.e., we simulate the two DFA's in the two state components of the product DFA.
- Make final states be pairs consisting of final states of both DFA's of L and M.

Product DFA: Example



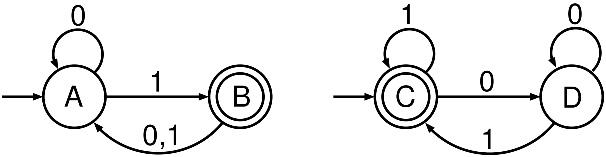


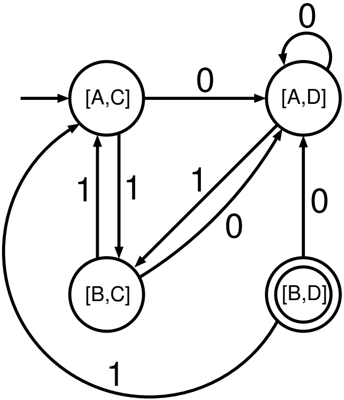


Closure Properties: Difference

- If L and M are regular, then so is $L - M$ = strings in L but not in M.
- **Proof:** Let A and B be DFA's whose languages are L and M.
- Construct the product DFA C of A and B.
- Make the final states of C be the pairs where A-state is final but B-state is not.

Difference: Example





Closure Properties: Containment

- If L and M are regular, then so is $L - M$ = strings in L but not in M.
- **Proof:** Let A and B be DFA's whose languages are L and M.
- Construct the product DFA C of A and B.
- Make the final states of C be the pairs where A-state is final but B-state is not.
- **Note:** Can also be used to test containment.
  - If $L - M = \emptyset$, then $L \subseteq M$.
  - How did we test if the language of a DFA is empty?

Closure Properties: Complement

- The complement of a language L is $\Sigma^* - L$.
- Since Σ^* is regular, the complement is always regular.

Closure Properties: Reversal

- Given language L, L^R has all strings whose reversal is in L.
- **Example:** $L = \{0, 01, 100\}$;
  $L^R = \{0, 10, 001\}$
- **Proof:** Let E be a regular expression for L.
- We show how to reverse E, to provide a regular expression E^R for L^R.

Reversal of a Regular Expression

- **Basis:** If E is a symbol a, ε, or $\emptyset$, then $E^R = E$.
- **Induction:** If E is:
  - $F + G$, then $E^R = F^R + G^R$
  - FG, then $E^R = G^R F^R$
  - F^*, then $E^R = (F^R)^*$

Reversal of a RE: Example

- Let $E = 01^* + 10^*$.

$$\begin{aligned} E^R &= (01^* + 10^*)^R \\ &= (01^*)^R + (10^*)^R \\ &= (1^*)^R 0^R + (0^*)^R 1^R \\ &= (1^{R*}) 0^R + (0^{R*}) 1^R \\ &= (1^*)0 + (0^*)1 \end{aligned}$$

Homomorphisms

- A homomorphism on an alphabet is a function that gives a string for each symbol in that alphabet.
- **Example:** $h(0) = ab$, $h(1) = \varepsilon$.
- Extend to strings by $h(a_1a_2 \ldots a_n)=h(a_1)h(a_2)\ldots h(a_n)$.
- **Example:** $h(01010) = ababab$.

Closure Properties: Homomorphism

- If L is regular, and h is a homomorphism on its alphabet, then $h(L) = \{h(w) \mid w \in L\}$ is also regular.
- **Proof:** Let E be a regular expression for L.
- Apply h to each symbol in E.
- Language of resulting RE is $h(L)$.

Closure under Homomorphism: Example

- Let $h(0) = ab$, $h(1) = \varepsilon$.
- Let L be the language of a regular expression $01^* + 10^*$.
- Then $h(L)$ is the language of regular expression $ab\varepsilon^* + \varepsilon(ab)^*$.
- **Note:** $ab\varepsilon^* + \varepsilon(ab)^* = ab + (ab)^* = (ab)^*$.