Equivalence of CFG's and PDA's

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Recap: Pushdown Automata

- A PDA is an automaton equivalent to the CFG in language-defining power.
- Only the nonterministic PDA's define all possible CFL's.
- But the deterministic version models parsers.
  - Most programming languages have deterministic PDA's.

Recap: Intuition

- Think of an ε-NFA with the additional power that it can manipulate a stack.
- Its moves are determined by:
  1. The current state (of its NFA).
  2. The current input symbol (or ε), and
  3. The current symbol on top of its stack.

Recap: Intuition

- Being nondeterministic, the PDA can have a choice of next moves.
- In each choice, the PDA can:
  1. Change state, and also
  2. Replace the top symbol on the stack by a sequence of zero or more symbols.
     - Zero symbols = pop.
     - Many symbols = sequence of pushes.

Recap: PDA Formalism

- A PDA is described by:
  1. A finite set of states (Q, typically).
  2. An input alphabet (Σ, typically).
  3. A stack alphabet (Γ, typically).
  4. A transition function (δ, typically).
  5. A start state (q_0, in Q, typically).
  6. A start symbol (Z_0, in Γ, typically).
  7. A set of final states ($F \subseteq Q$, typically).

Recap: The Transition Function

- Takes three arguments:
  1. A state in Q.
  2. An input which is either a symbol in Σ or ε.
  3. A stack symbol in Γ.
- $\delta(q,a,Z)$ is a set of zero or more actions of the form (p,α).
  - p is a state, α is a string of stack symbols.

Recap: Actions of the PDA

- If δ(q,a,Z) contains (p,α) among its actions, then one thing the PDA can do in state q, with a at the front of the input, and Z on top of the stack is:
    1. Change the state to p.
    2. Remove a from the front of the input (but a may be ε).
    3. Replace Z on the top of the stack by α.

Example: PDA

- Design a PDA to accept $\{0^n1^n \mid n \geq 1\}$.
- The states:
  - q = start state. We are in state q if we have seen only 0's so far.
  - p = we've seen at least one 1 and may now proceed only if the inputs are 1's.
  - f = final state; accept.

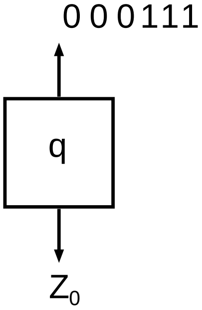
Example: PDA

- The stack symbols:
  - Z_0 = start symbol. Also marks the bottom of the stack, so we know we have counted the same number of 1's as 0's.
  - X = marker, used to count the number of 0's seen on the input.

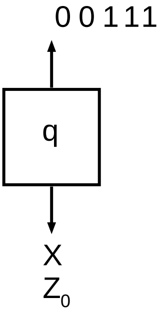
Example: PDA

- The transitions:
  - $\delta(q,0,Z_0) = \{(q,XZ_0)\}$.
  - $\delta(q,0,X) = \{(q,XX)\}$. These two rules cause one X to be pushed onto the stack for each 0 read from the input.
  - $\delta(q,1,X) = \{(p,\varepsilon)\}$. When we see a 1, go to state p and pop one X.
  - $\delta(p,1,X) = \{(p,\varepsilon)\}$. Pop one X per 1.
  - $\delta(p,\varepsilon,Z_0) = \{(f,Z_0)\}$. Accept at bottom.

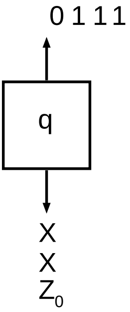
Actions of the Example PDA



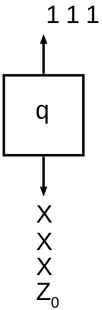
Actions of the Example PDA

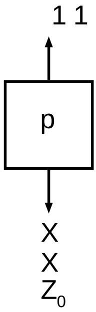


Actions of the Example PDA

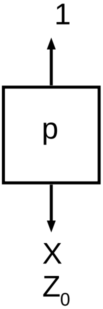


Actions of the Example PDA

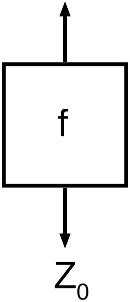



1 1
p
X
X
Z_0

Actions of the Example PDA



Actions of the Example PDA



Instantaneous Descriptions

- We can formalize the pictures just seen with an instantaneous description (ID).
- An ID is a triple (q,w,α), where:
  1. q is the current state.
  2. w is the remaining input.
  3. α is the stack contents, top at the left.

The "Goes-To" Relation

- To say that ID I can becomes ID J in one move of the PDA, we can write I $\vdash$ J.
- Formally, (q,aw,Xα) $\vdash$ (p,w,$\beta\alpha$) for any w and α, if δ(q,a,X) contains (p,β).
- Extend $\vdash$ to $\vdash^*$, meaning zero or more moves, by:
  - **Basis:** I $\vdash^*$ I.
  - **Induction:** If I $\vdash^*$ J and J $\vdash$ K, then I $\vdash^*$ K.

Example: Goes-To

- Using the previous example PDA, we can describe the sequence of moves by

$$\begin{aligned}(q, 000111, Z_0) &\vdash (q, 00111, XZ_0) \vdash (q, 0111, XXZ_0) \\ &\vdash (q, 111, XXXZ_0) \vdash (p, 11, XXZ_0) \\ &\vdash (p, 1, XZ_0) \vdash (p, \varepsilon, Z_0) \\ &\vdash (f, \varepsilon, Z_0)\end{aligned}$$

- Thus, $(q, 000111, Z_0) \vdash^* (f, \varepsilon, Z_0)$.

Example: Goes-To

- Using the previous example PDA, we can describe the sequence of moves by

$$
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&\vdash (p, 1, XZ_0) \vdash (p, \varepsilon, Z_0) \\
&\vdash (f, \varepsilon, Z_0)
\end{aligned}
$$

- Thus, $(\mathrm{q}, 000111, \mathrm{Z}_0) \vdash^* (\mathrm{f}, \varepsilon, \mathrm{Z}_0)$.

Question

What would happen on the input 0001111?

Answer

$$\begin{aligned}
(q, 0001111, Z_0) &\vdash (q, 001111, XZ_0) \vdash (q, 01111, XXZ_0) \\
&\vdash (q, 1111, XXXZ_0) \vdash (p, 111, XXZ_0) \\
&\vdash (p, 11, XZ_0) \vdash (p, 1, Z_0) \\
&\vdash (f, 1, Z_0)
\end{aligned}$$

- **Note:** The last action is legal because a PDA can use ε input even if input remains.
- The last ID has no move.
- 0001111 is not accepted, because the input is not completely consumed.

Aside: FA and PDA Notations

- We represented moves of a FA by an extended δ, which did not mention the input yet to be read.
- We could have chosen a similar notation for PDA's, where the FA state is replaced by a state-stack combination.

FA and PDA Notations

- Similarly, we could have chosen a FA notation with ID's.
  - Just drop the stack notation.
- Why the difference?
- FA tend to models thinks like protocols with infinitely long inputs.
- PDA model parsers, which are given a fixed program to process.

Language of a PDA

- The common way to define the language of a PDA is by final state.
- If P is a PDA, then L(P) is the set of strings w such that $(q_0, w, Z_0) \vdash^* (f, \varepsilon, \alpha)$ for final state f and any α.

Language of a PDA

- Another language defined by the same PDA is by empty stack.
- If P is a PDA, then $N(P)$ is the set of strings w such that $(q_0, w, Z_0) \vdash^* (q, \varepsilon, \varepsilon)$ for any state q.

Equivalence of Language Definitions

1. If $L = L(P)$, then there is another PDA P' such that
$$L = N(P')$$
2. If $L = N(P)$, then there is another PDA P'' such that
$$L = L(P'').$$

Proof: L(P) $\rightarrow$ N(P') Intuition

- P' will simulate P.
- If P accepts, P' will empty its stack.
- P' has to avoid accidentally emptying its stack, so it uses a special bottom marker to catch the case where P empties its stack without accepting.

Proof: L(P) $\rightarrow$ N(P')

- P' has all the states, symbols, and moves of P, plus:
  1. Stack symbol X_0, used to guard the stack bottom against accidental emptying.
  2. New start state s and erase state e.
  3. $\delta(s,\varepsilon,X_0) = \{(q_0,Z_0X_0)\}$. Get P started.
  4. $\delta(f,\varepsilon,X) = \delta(e,\varepsilon,X) = \{(e,\varepsilon)\}$ for any final state f of P and any stack symbol X.

Proof: N(P) $\rightarrow$ L(P”) Intuition

- P” simulates P.
- P” has a special bottom-marker to catch the situation where P empties its stack.
- If so, P” accepts.

Proof: N(P) → L(P")

- P" has all the states, symbols, and moves of P, plus:
  1. Stack symbol X_0, used to guard the stack bottom.
  2. New start state s and final state f.
  3. $\delta(s,\varepsilon,X_0) = \{(q_0,Z_0X_0)\}$. Get P started.
  4. $\delta(q,\varepsilon,X_0) = \{(f,\varepsilon)\}$ for any state q of P.

Deterministic PDA's

- To be deterministic, there must be at most one choice of move for any state q, input symbol a, and stack symbol X.
- In addition, there must not be a choice between using input ε or real input.
- Formally, δ(q,a,X) and δ(q,ε,X) cannot both be nonempty.

Equivalence of PDA's and CFG's: Overview

- When we talked about closure properties of regular languages, it was useful to be able to jump between RE and DFA representations.
- Similarly, CFG's and PDA's are both useful to deal with properties of CFL's.

Equivalence of PDA's and CFG's: Overview

- Also, PDA's, being algorithmic, are often easier to use when arguing that a language is a CFL.
- **Example:** It is easy to see how a PDA can recognize balanced parentheses, not so easy as a grammar.
- But all depends on knowing that CFG's and PDA's both define the CFL's.

Converting a CFG to a PDA

- Let $L = L(G)$.
- Construct PDA P such that $N(P) = L$.
- P has:
  - One state q.
  - Input symbols = terminals of G.
  - Stack symbols = all symbols of G.
  - Start symbol = start symbol of G.

Intuition about P

- Given input w, P will step through a leftmost derivation of w from the start symbol S.
- Since P can't know what this derivation is, or even what the end of w is, it uses nondeterminism to guess the production to use at each step.

Intuition about P

- At each step, P represents some left-sentential form (step of a leftmost derivation).
- If the stack of P is α, and P has so far consumed x from its input, then P represents left-sentential form $x\alpha$.
- At empty stack, the input consumed is a string in L(G).

Transition Function of P

1. δ(q,a,a) = (q,ε). (Type 1 rules)
   - This step does not change the LSF represented, but moves responsibility for a from the stack to the consumed input.
2. If A $\rightarrow$ α is a production of G, then δ(q,ε,A) contains (q,α). (Type 2 rules)
   - Guess a production for A, and represent the next LSF in the derivation.

Proof that N(P) = L(G)

- We need to show that (q,wx,S) $\vdash^*$ (q,x,α) for any x if and only if S $\Rightarrow^*_{\mathrm{lm}}$ wα.
- **Part 1:** only if is an induction on the number of steps made by P.
- **Basis:** 0 steps.
  - Then α = S, w = ε, and S $\Rightarrow^*_{\mathrm{lm}}$ S is surely true.

Induction for Part 1

- Consider n moves of P: (q,wx,S) $\vdash^*$ (q,x,α) and assume the **IH** for sequences of n-1 moves.
- There are two cases, depending on whether the last move uses a Type 1 or Type 2 rule.

Use of a Type 1 Rule

- The move sequence must be of the form $(q,yax,S) \vdash^{*} (q,ax,a\alpha) \vdash (q,x,\alpha)$, where $ya = w$.
- By the **IH** applied to the first n-1 steps, $S \Rightarrow^{*}_{lm} ya\alpha$.
- But $ya = w$, so $S \Rightarrow^{*}_{lm} w\alpha$.

Use of a Type 2 Rule

- The move sequence must be of the form (q,wx,S) $\vdash^*$ (q,x,Aβ) $\vdash$ (q,x,$\gamma\beta$), where A $\rightarrow$ γ is a production and $\alpha = \gamma\beta$.
- By the **IH** applied to the first n-1 steps, S $\Rightarrow^*_{lm}$ wAβ.
- Thus, S $\Rightarrow^*_{lm}$ w$\gamma\beta$ = wα.

Proof of Part 2 ("if")

- We also must prove that if $S \Rightarrow^*_{lm} w\alpha$, then $(q,wx,S) \vdash^* (q,x,\alpha)$ for any x.
- Induction on number of steps in the leftmost derivation.
- Ideas are similar.

Proof - Completion

- We now have $(q, wx, S) \vdash^* (q, x, \alpha)$ for any x if and only if $S \Rightarrow^*_{lm} w\alpha$.
- In particular, let $x = \alpha = \varepsilon$.
- Then $(q, w, S) \vdash^* (q, \varepsilon, \varepsilon)$ if and only if $S \Rightarrow^*_{lm} w$.
- That is, $w \in N(P)$ if and only if $w \in L(G)$.

From a PDA to a CFG

- Now assume $L = N(P)$.
- We'll construct a CFG G such that $L = L(G)$.
- **Intuition:** G will have variables generating exactly the inputs that cause P to have the net effect of popping a stack symbol X while going from state p to state q.
  - P never gets below this X while doing so.

Variables of G

- G's variables are of the form [pXq].
- This variable generates all and only the strings w such that

$$(p, w, X) \vdash^* (q, \varepsilon, \varepsilon)$$

- Also a start symbol S we'll talk about later.

Productions of G

- Each production for [pXq] comes from a move of P in state p with stack symbol X.
- **Simplest case:** δ(p,a,X) contains (q,ε).
- Then the production is [pXq] $\rightarrow$ a.
  - Note that a can be an input symbol or ε.
- Here, [pXq] generates a, because reading a is one way to pop X and go from p to q.

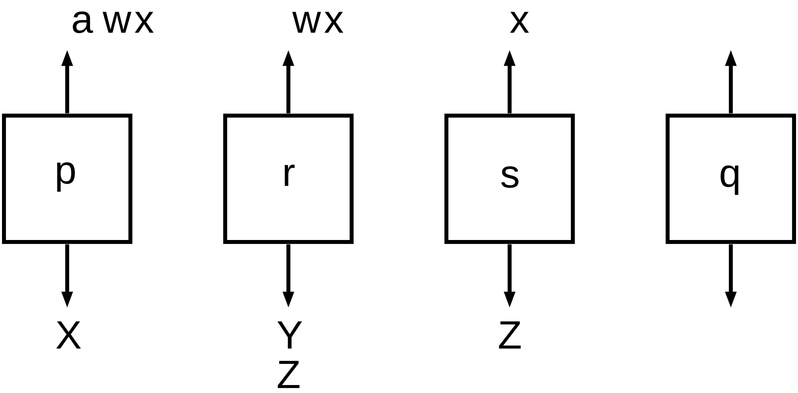
Productions of G

- **Next simplest case:** δ(p,a,X) contains (r,Y) for some state r and symbol Y.
- G has production [pXq] $\rightarrow$ a[rYq].
  - We can erase X and go from p to q by reading a (entering state r and replacing the X by Y) and then reading some w that gets P from r to q while erasing the Y.
- **Note:** [pXq] $\Rightarrow^*$ aw whenever [rYq] $\Rightarrow^*$ w.

Productions of G

- **Third simplest case:** δ(p,a,X) contains (r,YZ) for some state r and symbols Y and Z.
- Now, P has replaced X by YZ.
- To have the net effect of erasing X, P must erase Y, going from state r to some state s, and then erase Z, going from s to q.

Action of P



Productions of G

- Since we do not know state s, we must generate a family of productions:

$$[pXq] \rightarrow a[rYs][sZq]$$

- It follows $[pXq] \Rightarrow^* awx$ whenever $[rYs] \Rightarrow^* w$ and $[sZq] \Rightarrow^* x$.

Productions of G: General Case

- Suppose δ(p,a,X) contains (r,Y_1,...,Y_k) for some state r and k $\geq$ 3.
- Generate family of productions

$$[pXq] \rightarrow a[rY_1s_1][s_1Y_2s_2]\ldots[s_{k-2}Y_{k-1}s_{k-1}][s_{k-1}Y_kq]$$

Completion of the Construction

- We can prove that $(q_0,w,Z_0) \vdash^* (p,\varepsilon,\varepsilon)$ iff $[q_0Z_0p] \Rightarrow^* w$.
  - Proof is two easy inductions. Left as exercises.
- But state p can be anything.
- Thus, add to G another variable S, the start symbol, and add productions $S \rightarrow [q_0Z_0p]$ for each state p.