CS412: Introduction to Numerical Methods

MIDTERM #1 — 2:30PM - 3:45PM, Tuesday, 03/10/2015

Instructions: This exam is a **closed book** and **closed notes** exam, i.e., you are not allowed to consult any textbook, your class notes, homeworks, or any of the handouts from us. You are not permitted to refer to any other material either (including, of course, online material). No use of computers, cell phones, etc. is permitted.

Name	
University ID	

Part #1	
Part #2	
Part #3	
Part #4	
Part #5	
TOTAL	

1. [30% = 5 questions × 6% each] MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). No justification is required for your answer(s).

   (a) In scientific notation, $x = \pm a \times 2^b$, where a is the *mantissa* and b is the exponent. In single precision, we store $k = 23$ binary digits, and the exponent b ranges between $-126 \leq b \leq 127$. The largest number we can thus represent is:

      i. $(2 - 2^{-22}) \times 2^{127}$

      ii. $\boxed{(2 - 2^{-23}) \times 2^{127}}$

      iii. $(2 - 2^{-23}) \times 2^{126}$

   (b) The machine epsilon ε is defined as the smallest positive number that would actually achieve $1 + \varepsilon \neq 1$. With single precision numbers,

      i. $\boxed{\varepsilon = 2^{-23} \text{ (if by convention we always use truncation)}}$

      ii. $\varepsilon = 2^{-23}$ (if by convention we always use rounding)

      iii. $\boxed{\varepsilon = 2^{-24} \text{ (if by convention we always use rounding)}}$

   (c) In practice, Secant's method is preferred over Newton's method because:

      i. It is always guaranteed to converge, while Newton's method is not.

      ii. [It is computationally cheaper than Newton's method.]

      iii. [It does not require computing the derivative.]

   (d) In practice, the order of convergence of Newton's method is:

      i. always 2, if f does not have a multiple root.

      ii. [somewhere between 1 and 2, if f does not have a multiple root, because it is not always guaranteed to converge and so is used in combination with bisection search.]

      iii. [1 if the unknown function f has a multiple root and its explicit formula is known.]

   (e) Which of the following are valid reasons for using piecewise polynomial interpolation, as opposed to using a single polynomial?

      i. [In the monomial basis, piecewise polynomials are cheaper to compute than high-degree polynomials.]

      ii. Piecewise polynomials can be extended to include more data points, while it is impossible to update a single polynomial interpolant incrementally to include additional points.

      iii. [High-degree polynomials can suffer from global changes when a single data point is perturbed, while piecewise polynomials only change locally.]

2. [24% = 3 questions × 8% each] SHORT ANSWER SECTION. Answer each of the following questions in no more than 2-3 sentences.

   (a) Even if the cost of solving a linear system was cheap, what would be a good reason to still not use Vandermonde interpolation?

   Answer: Higher degree monomials start looking very similar to each other, and so with limited precision, the matrix V starts becoming close to singular even though it is non-singular with exact arithmetic.

   (b) Describe two benefits of using Chebyshev points for polynomial interpolation.

   Answer:

   - It ensures that the polynomial interpolant will converge to the function $f(x)$ being sampled as more data points are added, provided f and its first derivative are bounded.
   - It drastically reduces the risk of oscillatory interpolants associated with using high order polynomials.

   (c) Give an example to show that Hermite interpolation does not guarantee continuity of the second derivative between piecewise polynomials.

   Answer: Choose $x_1 = -1, x_2 = 0, x_3 = 1$. Let $y_1 = 1, y_1' = 2, y_2 = y_2' = 0, y_3 = 1, y_3' = 3$. For the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, choose $s_1(x)$ defined over $[x_1, x_2]$ as $s_1(x) = x^2$, and choose $s_2(x)$ defined over $[x_2, x_3]$ as $s_2(x) = x^3$. Clearly, the piecewise polynomial $s(x)$ defined as

$$s(x) = \begin{cases} s_1(x), x \in [x_1, x_2] \\ s_2(x), x \in [x_2, x_3] \end{cases}$$

   is a cubic Hermite spline. However, $s_1''(0) = 2$, while $s_2''(0) = 0$, showing that the second derivatives are not continuous.

3. [14%] Use Lagrange interpolation to find a cubic polynomial in the monomial basis that interpolates the following four data points:

$$(-4,-3), \quad (-2,1), \quad (0,2), \quad (1,1)$$

Reminder: Lagrange polynomials are given by the formula:

$$l_i(x) = \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j\neq i}(x_i - x_j)}$$

Answer:

$$s(x) = -\frac{1}{40}x^3 - \frac{21}{40}x^2 - \frac{9}{20}x + 2$$

4. [12%] Using any of the methods we discussed in class, find a cubic polynomial $s(x)$ in the monomial basis, defined over $[0, 1]$, that satisfies:

$$\begin{aligned} s(0) &= 1 \\ s'(0) &= -1 \\ s(1) &= 2 \\ s'(1) &= -3 \end{aligned}$$

Note: In case you decide to use the Hermite basis polynomials, those are given below:

$$\begin{aligned} h_{00}(x) &= 2x^3 - 3x^2 + 1 \\ h_{01}(x) &= -2x^3 + 3x^2 \\ h_{10}(x) &= x^3 - 2x^2 + x \\ h_{11}(x) &= x^3 - x^2 \end{aligned}$$

Answer:

$$s(x) = -6x^3 + 8x^2 - x + 1$$

5. [20%] Consider a piecewise *quadratic* polynomial function $s(x)$ that interpolates the n data points $(x_1, y_1), \ldots, (x_n, y_n)$. In each subinterval $I_k = [x_k, x_{k+1}]$, we define $s(x)$ as a quadratic polynomial $s_k(x) = a_2^{(k)} x^2 + a_1^{(k)} x + a_0^{(k)}$, such that:

   - For $k = 1, 2, \ldots, n-2$, $s_k(x)$ should interpolate points (x_k, y_k), (x_{k+1}, y_{k+1}) and (x_{k+2}, y_{k+2}),
   - $s_{n-1}(x)$ should interpolate (x_{n-2}, y_{n-2}), (x_{n-1}, y_{n-1}) and (x_n, y_n).

   Derive a bound for the interpolation error $|f(x) - s(x)|$. You may assume that the length $h_k = |x_{k+1} - x_k|$ of each subinterval is constant and equal to h.

   Note: You may find the following theorem useful.

   Theorem 1. *Let*

   - $x_0 < x_1 < \ldots < x_{n-1} < x_n$
   - $y_n = f(x_n)$, $k = 0, 1, \ldots, n$, *where* f *is a function which is* n*-times differentiable with continuous derivatives*
   - $\mathcal{P}_n(x)$ *is a polynomial that interpolates* $(x_0, y_0), (x_1, y_1) \ldots, (x_n, y_n)$

   then for any $x \in (x_0, x_n)$, *there exists a* $\theta = \theta(x) \in (x_0, x_n)$ *such that*

$$f(x) - \mathcal{P}_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n)$$

   Answer: From the given theorem, it follows that

$$|f(x) - s_k(x)| = \frac{f^{(3)}(\theta)}{3!} \underbrace{(x - x_k)(x - x_{k+1})(x - x_{k+2})}_{q(x)}$$

   We need to compute the maximum value that $q(x)$ can take in the interval $[x_k, x_{k+1}]$. Clearly, this value will always be less (or equal to) the maximum value that $q(x)$ can take in the interval $[x_k, x_{k+2}]$. Now,

$$\begin{aligned} q(x) &= x^3 - (x_k + x_{k+1} + x_{k+2})x^2 + (x_k x_{k+1} + x_{k+1} x_{k+2} + x_k x_{k+2})x - x_k x_{k+1} x_{k+2} \\ \Rightarrow q'(x) &= 3x^2 - 2(x_k + x_{k+1} + x_{k+2})x + (x_k x_{k+1} + x_{k+1} x_{k+2} + x_k x_{k+2}) \end{aligned}$$

   The maximum value of $q(x)$ will be attained at that root of $q'(x)$ where $q''(x) < 0$. This root comes out to be:

$$x = \frac{1}{3}\left\{(x_k + x_{k+1} + x_{k+2}) - \sqrt{(x_k + x_{k+1} + x_{k+2})^2 - 3(x_k x_{k+1} + x_{k+1} x_{k+2} + x_k x_{k+2})}\right\}$$

   The discriminant above can be written as:

$$\begin{aligned} D &= (x_k + x_{k+1} + x_{k+2})^2 - 3(x_k x_{k+1} + x_{k+1} x_{k+2} + x_k x_{k+2}) \\ &= \frac{1}{2}(x_k - x_{k+1})^2 + \frac{1}{2}(x_{k+1} - x_{k+2})^2 + \frac{1}{2}(x_k - x_{k+2})^2 \\ &= \frac{h^2}{2} + \frac{h^2}{2} + \frac{4h^2}{2} = 3h^2 \end{aligned}$$

Substituting these expression into $q(x)$ gives,

$$q(x) \leq \left\{ \frac{(x_{k+1}+x_{k+2}-2x_k)}{3} - \frac{\sqrt{3h^2}}{3} \right\} \left\{ \frac{(x_k+x_{k+2}-2x_{k+1})}{3} - \frac{\sqrt{3h^2}}{3} \right\} \left\{ \frac{(x_k+x_{k+1}-2x_{k+2})}{3} - \frac{\sqrt{3h^2}}{3} \right\}$$

Simplifying the above equation by using the relations

$$\begin{aligned} x_{k+1}+x_{k+2}-2x_k &= x_{k+1}-x_k+x_{k+2}-x_k = h+2h = 3h \\ x_k+x_{k+2}-2x_{k+1} &= x_k-x_{k+1}+x_{k+2}-x_{k+1} = -h+h = 0 \\ x_k+x_{k+1}-2x_{k+2} &= x_k-x_{k+2}+x_{k+1}-x_{k+2} = -2h-h = -3h \end{aligned}$$

gives

$$\begin{aligned} q(x) &\leq \left\{ \frac{3h}{3} - \frac{\sqrt{3h^2}}{3} \right\} \left\{ 0 - \frac{\sqrt{3h^2}}{3} \right\} \left\{ -\frac{3h}{3} - \frac{\sqrt{3h^2}}{3} \right\} \\ &\leq \left\{ h^2 - \frac{h^2}{3} \right\} \frac{h}{\sqrt{3}} = \frac{2h^3}{3\sqrt{3}} \end{aligned}$$

Using this relation over the entire interval $[x_1, x_n]$ gives

$$|f(x) - s(x)| \leq \frac{1}{6}||f^{(3)}||_\infty \cdot \frac{2h^3}{3\sqrt{3}} = \frac{1}{9\sqrt{3}}||f^{(3)}||_\infty \cdot h^3$$