CS412 Spring Semester 2011

Practice Midterm #2 – Solutions

1. MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). You do not need to provide a justification for your answer(s).

   (1) Numerical integration rules which use just one point per interval are:
   (Circle or underline the ONE most correct answer)

   (a) Always first order accurate. Methods that use 2 points are second order accurate and so on.

   (b) At most first order accurate, but can never have an order of accuracy of 2 or more.

   (c) $\boxed{\text{Generally first order accurate, but sometimes could also be second}}$ $\boxed{\text{order due to fortuitious cancellation.}}$

   Comments: The rectangle rule is first order, while the midpoint rule is second order. Both use just a single point

   (2) A certain numerical integration rule integrates all cubic polynomials exactly. Which of the following property describes its accuracy?
   (Circle or underline the ONE most correct answer)

   (a) The method is exactly third order accurate.

   (b) $\boxed{\text{The method is at least fourth order accurate.}}$

   (c) The global error is proportional to h^3.

   Comments: When any polynomial of up to $n-1$ degree is integrated exactly, the method is at least n-order accurate. This also means the global error scales like $O(h^n)$, the local like $O(h^{n+1})$.

   (3) Which of the following methods are good choices for solving $\mathbf{Ax} = \mathbf{b}$, where A is symmetric and positive definite?
   (Circle or underline ALL correct answers)

   (a) **LU** factorization with full pivoting.

   (b) **QR** factorization.

   (c) System of normal equations.

   (d) $\boxed{\text{Gauss-Seidel method.}}$

   (e) Jacobi method.

   Comments: Even if it were a viable choice, LU factorization would not require pivoting on a symmetric, positive definite matrix. (b,c) are for least squares problems. The Jacobi method requires diagonal dominance to guarantee convergence.

(4) Why would we ever use an explcit method instead of an implicit one for solving an Initial Value Problem?
(Circle or underline ALL correct answers)

(a) Explicit methods are more robust for ODEs with unstable solutions.

(b) $\boxed{\text{Explicit methods do not require solving a nonlinear system.}}$

(c) $\boxed{\text{If we are willing to use a small enough time step } dt\text{, each iteration of an explicit method is quite cheap.}}$

(d) Explicit methods are unconditionally stable.

Comments: When the conditions for stability are satisfied, explicit methods are typically cheaper. However, they are generally the least stable alternatives.

2. SHORT ANSWER SECTION. Answer each of the following questions in no more than 1-2 sentences.

(a) Describe one scenario where you would prefer using the **LU** factorization, instead of an iterative method such as Jacobi or Gauss-Seidel, for solving a linear system $\mathbf{Ax} = \mathbf{b}$.

When we have many systems $\mathbf{Ax}_k = \mathbf{b}_k$ to solve, with the same coefficient matrix. Or when the matrix is neither symmetric, positive definite or diagonally dominant. It can also be a viable method when the size of the system is very small.

(b) What is one benefit of the **QR** factorization when compared to the normal equations, as methods for solving least squares problems?

The QR factorization does not square the condition number of the system, while the normal equations do. Thus, the QR approach is better conditioned..

(c) Describe one reason why solving a system $\mathbf{Ax} = \mathbf{b}$ could be extremely challenging when $\mathbf{A}$ has a very high condition number.

Because tiny inaccuracies in the right hand side, or the solution methodology can translate to gigantic errors in the computed solution.

(d) Describe one valid reason for using Forward Euler, instead of Backward Euler to solve an initial value problem. Also, what would be a reason for choosing Backward Euler in this case?

Forward Euler incurs very little computation per iteration; thus when the time step Δt *is small enough to guarantee stability, Forward Euler is very cheap. On the other hand, Backward Euler is unconditionally stable, while Forward Euler places restrictions on* Δt *for stability.*

(e) Why is Simpson's rule potentially much more attractive than the trapezoidal rule, when approximating definite integrals?

For a small, almost negligible increase in algorithmic complexity, Simpson's rule offers 4th order accuracy, as opposed to the 2nd order accurate Trapezoidal Rule.

(f) Describe one plausible stopping criterion for determining when to stop an iterative solver for $\mathbf{Ax} = \mathbf{b}$, such as Jacobi or Gauss-Seidel.

$\|\mathbf{b} - \mathbf{A}\mathbf{x}_k\|$=*small. Or* $\|\mathbf{x}_{k+1} - \mathbf{x}_k\|$=*small.*

3. Show the following properties:

   (a) $\|\mathbf{x}\|_2^2 = \mathbf{x}^T\mathbf{x}$ for any vector $\mathbf{x} \in \mathbf{R}$.

   (b) $\|\mathbf{Q}\|_2 = 1$, for any *orthogonal* matrix $\mathbf{Q}$.

   Solution:

   (a)

$$\|\mathbf{x}\|_2^2 = \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i \cdot x_i = \mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T\mathbf{x}$$

   (b)

$$\|\mathbf{Qx}\|_2^2 = (\mathbf{Qx})^T\mathbf{Qx} = \mathbf{x}^T\mathbf{Q}^T\mathbf{Qx} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|_2^2$$

   Thus, $\|\mathbf{Qx}\|_2 = \|\mathbf{x}\|_2$, and

$$\|\mathbf{Q}\|_2 = \max \frac{\|\mathbf{Qx}\|_2}{\|\mathbf{x}\|_2} = \max \frac{\|\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = 1$$

4. Consider the $n > 3$ data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. We want to find a cubic polynomial $p(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ such that the graph of $p(x)$ approximates the given data points as much as possible. Write a Least Squares system $\mathbf{Ax} \approx \mathbf{b}$ which can be used to determine this cubic approximating polynomial. What does this system reduce to, in the case $n = 4$?

   Solution:

   We want to approximate:

$$c_3x_1^3 + c_2x_1^2 + c_1x_1 + c_0 \approx y_1$$
$$c_3x_2^3 + c_2x_2^2 + c_1x_2 + c_0 \approx y_2$$
$$\vdots$$
$$c_3x_n^3 + c_2x_n^2 + c_1x_n + c_0 \approx y_n$$

   Or in matrix form:

$$\begin{pmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n^3 & x_n^2 & x_n & 1 \end{pmatrix} \begin{pmatrix} c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} \approx \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$$

   Which is an $n \times 4$ least squares system $\mathbf{Ax} \approx \mathbf{b}$. When $n = 4$ the system becomes *square* (4×4) and an exact solution is obtainable. This is now simply the Vandermonde system for polynomial interpolation.

5. Show that the coefficient matrix in the system of normal equations (used for solving least squares problems) is always symmetric and positive definite.

   Solution:

   The system of normal equations is $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$. We need to show that $\mathbf{A}^T\mathbf{A}$ is symmetric and positive definite. The symmetry is easy to show, since

$$\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T\left(\mathbf{A}^T\right)^T = \mathbf{A}^T\mathbf{A}.$$

   For positive definiteness, we need to show, that $\mathbf{x}^T\mathbf{A}^T\mathbf{Ax} \geq 0$ for all $\mathbf{x} \in \mathbf{R}$. We have in fact:

$$\mathbf{x}^T\mathbf{A}^T\mathbf{Ax} = (\mathbf{Ax})^T(\mathbf{Ax}) = \|\mathbf{Ax}\|_2^2 \geq 0$$

6. Consider the $n \times n$ linear system $\mathbf{Ax} = \mathbf{b}$.

   (a) Show that if $\mathbf{A}$ is diagonal, the Jacobi method converges after just one iteration.

   (b) Show that if $\mathbf{A}$ is lower triangular, the Gauss-Seidel method converges after just one iteration.

   Solution:

   In both cases, we start by considering the decomposition $\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$. Also let us denote the exact solution by $\mathbf{x}^*$, i.e. $\mathbf{Ax}^* = \mathbf{b}$.

   (a) the first iteration of Jacobi's method will give:

   $$\mathbf{Dx}^{(1)} = (\mathbf{L} + \mathbf{U})\mathbf{x}^{(0)} + \mathbf{b}$$

   Since $\mathbf{A}$ is diagonal, we have $\mathbf{A} = \mathbf{D}$ and $\mathbf{U} = \mathbf{L} = \mathbf{0}$. Thus, the previous equation becomes:

   $$\begin{aligned} \mathbf{Ax}^{(1)} &= \mathbf{0} \cdot \mathbf{x}^{(0)} + \mathbf{b} \\ \mathbf{Ax}^{(1)} &= \mathbf{Ax}^* \\ \mathbf{x}^{(1)} &= \mathbf{x}^* \end{aligned}$$

   (b) the first iteration of Gauss-Seidel will give:

   $$(\mathbf{D} - \mathbf{L})\mathbf{x}^{(1)} = \mathbf{Ux}^{(0)} + \mathbf{b}$$

   Since $\mathbf{A}$ is lower triangular, we have $\mathbf{A} = \mathbf{D} - \mathbf{L}$ and $\mathbf{U} = \mathbf{0}$. Thus, the previous equation becomes:

   $$\begin{aligned} \mathbf{Ax}^{(1)} &= \mathbf{0} \cdot \mathbf{x}^{(0)} + \mathbf{b} \\ \mathbf{Ax}^{(1)} &= \mathbf{Ax}^* \\ \mathbf{x}^{(1)} &= \mathbf{x}^* \end{aligned}$$

7. The numerical integration rule known as *Simpson's 3/8 Rule* is defined as:

   $$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$

   (a) Determine the order of accuracy of this method.

   (b) Describe a composite rule based on the formula above.

Solution:

We start testing this rule on monomials of the form $f(x) = x^d$. We have:

- $f(x) = 1$:

$$I_{\text{rule}} = \frac{b-a}{8}\left[1+3+3+1\right] = b-a \equiv \int_a^b 1 \cdot dx$$

- $f(x) = x$:

$$I_{\text{rule}} = \frac{b-a}{8}\left[a + 3\frac{2a+b}{3} + 3\frac{a+2b}{3} + b\right] = \frac{b^2}{2} - \frac{a^2}{2} \equiv \int_a^b x dx$$

- $f(x) = x^2$:

$$I_{\text{rule}} = \frac{b-a}{8}\left[a^2 + 3\left(\frac{2a+b}{3}\right)^2 + 3\left(\frac{a+2b}{3}\right)^2 + b^2\right] =$$

$$= \frac{b^3}{3} - \frac{a^3}{3} \equiv \int_a^b x^2 dx$$

- $f(x) = x^3$:

$$I_{\text{rule}} = \frac{b-a}{8}\left[a^3 + 3\left(\frac{2a+b}{3}\right)^3 + 3\left(\frac{a+2b}{3}\right)^3 + b^3\right] =$$

$$= \frac{b^4}{4} - \frac{a^4}{4} \equiv \int_a^b x^3 dx$$

- $f(x) = x^4$:

$$I_{\text{rule}} = \frac{b-a}{8}\left[a^4 + 3\left(\frac{2a+b}{3}\right)^4 + 3\left(\frac{a+2b}{3}\right)^4 + b^4\right] =$$

$$= \frac{b-a}{54}\left[11a^4 + 10a^3b + 12a^2b^2 + 10ab^3 + 11b^4\right] \neq \int_a^b x^4 dx$$

Since this rule integrates up to cubic poynomials exactly, it is fourth-order accurate.

In order to generate a composite rule, we write this method using 4 consequtive points $(x_k, x_{k+1}, x_{k+2}, x_{k+3})$, as follows:

$$\int_{x_k}^{x_{k+3}} f(x)dx \approx \frac{3h}{8}\left[f(x_k) + 3f\left(x_{k+1}\right) + 3f\left(x_{k+2}\right) + f(x_{k+3})\right]$$

The corresponding composite rule over points $x_0, x_1, x_2, \ldots, x_{3n}$ is generated by applying the rule above to compute the integral over each subsequent span of 3 subintervals. Thus:

$$\int_{x_0}^{x_{3n}} f(x)dx = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+3}} f(x)dx \approx$$

$$\approx \sum_{k=0}^{n-1} \frac{3h}{8} \left[f(x_k) + 3f\left(x_{k+1}\right) + 3f\left(x_{k+2}\right) + f(x_{k+3}) \right].$$

8. The numerical integration rule known as *Milne's Rule* is defined as:

$$\int_a^b f(x)dx \approx \frac{b-a}{3} \left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right) \right]$$

Determine the order of accuracy of this method.

We start testing this rule on monomials of the form $f(x) = x^d$. We have:

- $f(x) = 1$:

$$I_{\text{rule}} = \frac{b-a}{3} [2 - 1 - 2] = b - a \equiv \int_a^b 1 \cdot dx$$

- $f(x) = x$:

$$I_{\text{rule}} = \frac{b-a}{3} \left[2\left(\frac{3a+b}{4}\right) - \left(\frac{a+b}{2}\right) + 2\left(\frac{a+3b}{4}\right) \right] =$$

$$= \frac{b^2}{2} - \frac{a^2}{2} \equiv \int_a^b x dx$$

- $f(x) = x^2$:

$$I_{\text{rule}} = \frac{b-a}{3} \left[2\left(\frac{3a+b}{4}\right)^2 - \left(\frac{a+b}{2}\right)^2 + 2\left(\frac{a+3b}{4}\right)^2 \right] =$$

$$= \frac{b^3}{3} - \frac{a^3}{3} \equiv \int_a^b x dx$$

- $f(x) = x^3$:

$$I_{\text{rule}} = \frac{b-a}{3} \left[2\left(\frac{3a+b}{4}\right)^3 - \left(\frac{a+b}{2}\right)^3 + 2\left(\frac{a+3b}{4}\right)^3 \right] =$$

$$= \frac{b^4}{4} - \frac{a^4}{4} \equiv \int_a^b x dx$$

- $f(x) = x^4$:

$$I_{\text{rule}} = \frac{b-a}{3}\left[2\left(\frac{3a+b}{4}\right)^4 - \left(\frac{a+b}{2}\right)^4 + 2\left(\frac{a+3b}{4}\right)^4\right] =$$

$$= \frac{b-a}{192}\left[37a^4 + 44a^3b + 30a^2b^2 + 44ab^3 + 37b^4\right] \neq \int_a^b x^4 dx$$

Since this rule integrates up to cubic poynomials exactly, it is fourth-order accurate.

9. Consider a numerical integration rule defined as:

$$\int_a^b f(x)dx \approx w_1 f(a) + w_2 f\left(\frac{2a+b}{3}\right) + w_3 f(b)$$

where w_1, w_2, w_3 are undetermined constants. Find the values of w_1, w_2, w_3 such that this rule becomes 3rd order accurate.

Solution:

For 3rd order accuracy, this rule needs to integrate exactly the monomials $1, x$ and x^2. Thus, we have:

- $f(x) = 1$:

$$I_{\text{rule}} = w_1 + w_2 + w_3$$

Thus we need:

$$w_1 + w_2 + w_3 = b - a \tag{1}$$

- $f(x) = x$:

$$I_{\text{rule}} = aw_1 + \left(\frac{2a+b}{3}\right) w_2 + bw_3$$

Thus we need:

$$aw_1 + \left(\frac{2a+b}{3}\right) w_2 + bw_3 = \frac{b^2}{2} - \frac{a^2}{2} \tag{2}$$

- $f(x) = x^2$:

$$I_{\text{rule}} = a^2 w_1 + \left(\frac{2a+b}{3}\right)^2 w_2 + b^2 w_3$$

Thus we need:

$$a^2 w_1 + \left(\frac{2a+b}{3}\right)^2 w_2 + b^2 w_3 = \frac{b^3}{3} - \frac{a^3}{3} \tag{3}$$

Equations (1,2,3) can be solved for the values of w_1, w_2, w_3 (the solution would be considered correct if stopped here). The exact solutions are $w_1 = 0, w_2 = 3(b-a)/4, w_3 = (b-a)/4$.

10. Consider the ordinary differential equation

$$y'(t) = f(t, y). \tag{4}$$

(a) If the solutions to equation (4) are stable, show that the solutions of

$$z'(t) = f(t, z) + g(t) \tag{5}$$

are also stable ($g(t)$ is an arbitrary function). Furthermore, if the solutions to equation (4) are asymptotically stable, show that so will be the solutions to the differential equation (5).

(b) Consider the special case $f(t, z) = \lambda z$, $\lambda < 0$. Show that any function of the form

$$z(t) = ce^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda \tau} g(\tau) d\tau$$

is a solution of equation (5). If we assume that these are *all* the solutions to equation (5), can you show directly (without using the derivative criterion) that they are asymptotically stable?

Solution:

(a) We have:

$$\frac{\partial}{\partial y} f(t, y) \equiv \frac{\partial}{\partial y} \{f(t, y) + g(t)\}$$

Thus, the partial derivatives (with respect to y) of the right-hand sides of both ODEs are equal; As a consequence, if one is stable (or asymptotically stable), the other will be too.

(b) Let $y_1(t), y_2(t)$ be the two solutions of the ODE that correspond to values c_1 and c_2 of the undetermined constant. Using the formula for these solutions, we have that

$$|y_1(t) - y_2(t)| = |c_1 e^{\lambda t} - c_2 e^{\lambda t}| = |c_1 - c_2| e^{\lambda t} \overset{t \to \infty}{\longrightarrow} 0$$

since $\lambda < 0$. Thus, by definition, the solutions are asymptotically stable.

11. We have seen that many 1-step methods for initial value problems are created by integrating the ODE as follows:

$$y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y) d\tau$$

and then approximating the integral on the right hand side by a numerical integration rule.

Describe the method that results from using Simpson's rule for approximating this integral, and determine its stability condition on the model equation $y' = \lambda y, \ \lambda < 0$.

Solution:

Applying Simpson's rule to approximate the integral, we obtain:

$$y_{k+1} - y_k = \frac{t_{k+1} - t_k}{6}\left[f(t_k, y_k) + 4f(\frac{t_k + t_{k+1}}{2}, \frac{y_k + y_{k+1}}{2}) + f(t_{k+1}, y_{k+1})\right]$$

$$y_{k+1} = y_k + \frac{\Delta t}{6}\left[f(t_k, y_k) + 4f(\frac{t_k + t_{k+1}}{2}, \frac{y_k + y_{k+1}}{2}) + f(t_{k+1}, y_{k+1})\right]$$

For the model ODE $y' = \lambda y,\ \lambda < 0$ we substitute $f(t, y) = \lambda y$ to obtain:

$$y_{k+1} = y_k + \frac{\Delta t}{6}\left[\lambda y_k + 4\lambda\frac{y_k + y_{k+1}}{2} + \lambda y_{k+1}\right]$$

$$y_{k+1} = y_k + \frac{\Delta t}{2}\left[\lambda y_k + \lambda y_{k+1}\right]$$

This is exactly the same as the Trapezoidal rule (specifically, for this model ODE). Thus, we have that the method is unconditionally stable (following the exact same proof from this point, as we did for Trapezoidal rule).