CS412: Homework #3

Due: Tuesday, March 24th, 2015 (by 11:59PM)

Sum of all problems: 120%, Maximum possible score: 100%.

1. [40%] Prove the following properties:

   (a) Show that for any vector $x \in \mathbb{R}^n$, the following inequalities hold:

   $$||x||_\infty \leq ||x||_1 \leq n||x||_\infty$$
   $$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$$

   (b) Assume that positive constants c_1, c_2 exist, such that for any $x \in \mathbb{R}^n$

   $$c_1||x||_a \leq ||x||_b \leq c_2||x||_a$$

   Here, $||\cdot||_a$ and $||\cdot||_b$ are simply two different vector norms. Show that in this case, we can also find positive constants d_1, d_2 such that

   $$d_1||M||_a \leq ||M||_b \leq d_2||M||_a$$

   for any *matrix* $M \in \mathbb{R}^{n\times n}$. The norms in the last expression are the matrix norms induced from the respective vector norms.

2. [20%] Let

   $$A = \begin{bmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix}$$

   (a) What is the determinant of A?

   (b) In single-precision arithmetic, for what range of values of ε will the computed value of the determinant be zero?

   (c) What is the LU factorization of A?

   (d) In single-precision arithmetic, for what range of values of ε will the computed value of U be singular?

3. [20%] Prove the following, where A, U, V are $n \times n$ matrices and u, v are $n \times 1$ vectors:

   (a) The Sherman-Morrison formula:

   $$(A - uv^T)^{-1} = A^{-1} + A^{-1}u(1 - v^T A^{-1}u)^{-1}v^T A^{-1}$$

   Hint: Multiply both sides by $(A - uv^T)$.

   (b) The Woodbury formula:

   $$(A - UV^T)^{-1} = A^{-1} + A^{-1}U(I - V^T A^{-1}U)^{-1}V^T A^{-1}$$

   Hint: Multiply both sides by $(A - UV^T)$.

4. [40%] Prove the following two statements:

   (a) The product of two lower triangular matrices is lower triangular.

   (b) The inverse of a nonsingular lower triangular matrix is lower triangular.