1. [40%] Prove the following properties:
   (a) Show that for any vector $x \in \mathbb{R}^n$, the following inequalities hold:
   \[
   ||x||_\infty \leq ||x||_1 \leq n ||x||_\infty \\
   ||x||_\infty \leq ||x||_2 \leq \sqrt{n} ||x||_\infty
   \]
   (b) Assume that positive constants $c_1, c_2$ exist, such that for any $x \in \mathbb{R}^n$
   \[
   c_1 ||x||_a \leq ||x||_b \leq c_2 ||x||_a
   \]
   Here, $|| \cdot ||_a$ and $|| \cdot ||_b$ are simply two different vector norms. Show that in this case, we can also find positive constants $d_1, d_2$ such that
   \[
   d_1 ||M||_a \leq ||M||_b \leq d_2 ||M||_a
   \]
   for any matrix $M \in \mathbb{R}^{n \times n}$. The norms in the last expression are the matrix norms induced from the respective vector norms.

2. [20%] Let
   \[
   A = \begin{bmatrix}
   1 & 1 + \varepsilon \\
   1 - \varepsilon & 1
   \end{bmatrix}
   \]
   (a) What is the determinant of $A$?
   (b) In single-precision arithmetic, for what range of values of $\varepsilon$ will the computed value of the determinant be zero?
   (c) What is the $LU$ factorization of $A$?
   (d) In single-precision arithmetic, for what range of values of $\varepsilon$ will the computed value of $U$ be singular?
3. [20%] Prove the following, where $A, U, V$ are $n \times n$ matrices and $u, v$ are $n \times 1$ vectors:

(a) The Sherman-Morrison formula:

$$(A - uv^T)^{-1} = A^{-1} + A^{-1}u(1 - v^TA^{-1}u)^{-1}v^TA^{-1}$$

*Hint:* Multiply both sides by $(A - uv^T)$.

(b) The Woodbury formula:

$$(A - UV^T)^{-1} = A^{-1} + A^{-1}U(I - V^TA^{-1}U)^{-1}V^TA^{-1}$$

*Hint:* Multiply both sides by $(A - UV^T)$.

4. [40%] Prove the following two statements:

(a) The product of two lower triangular matrices is lower triangular.

(b) The inverse of a nonsingular lower triangular matrix is lower triangular.