CS412: Lecture #11

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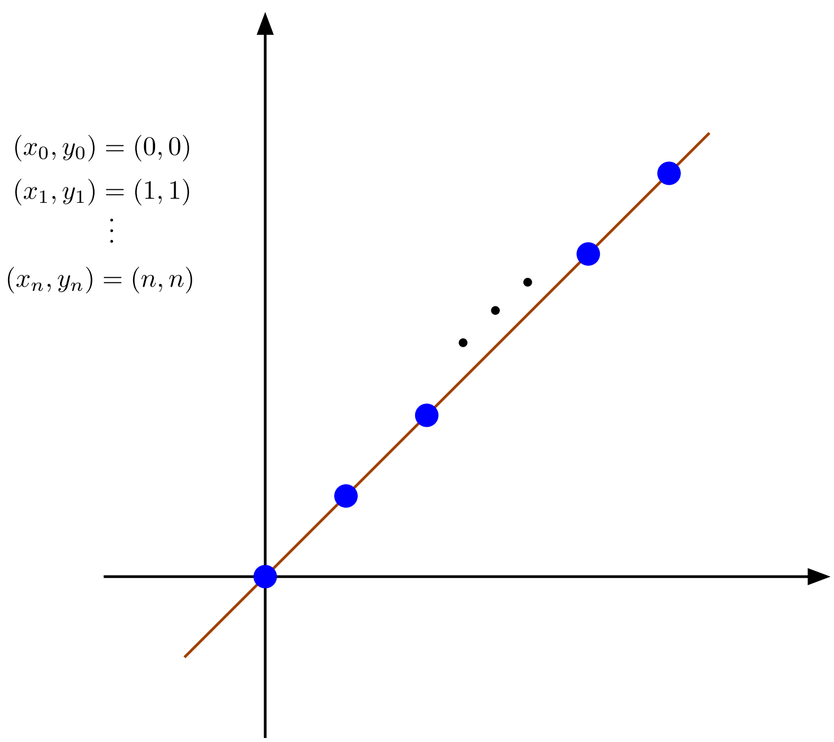
We saw three methods for polynomial interpolation (Vandermonde, Lagrange, Newton). It is important to understand that all three methods compute (in theory) the *same exact interpolant* $\mathcal{P}_n(x)$, just following different paths which may be better or worse from a computational perspective. The question however remains:

- How accurate is this interpolation, or in other words,
- How close is $\mathcal{P}_n(x)$ to the "real" function $f(x)$?

Example:



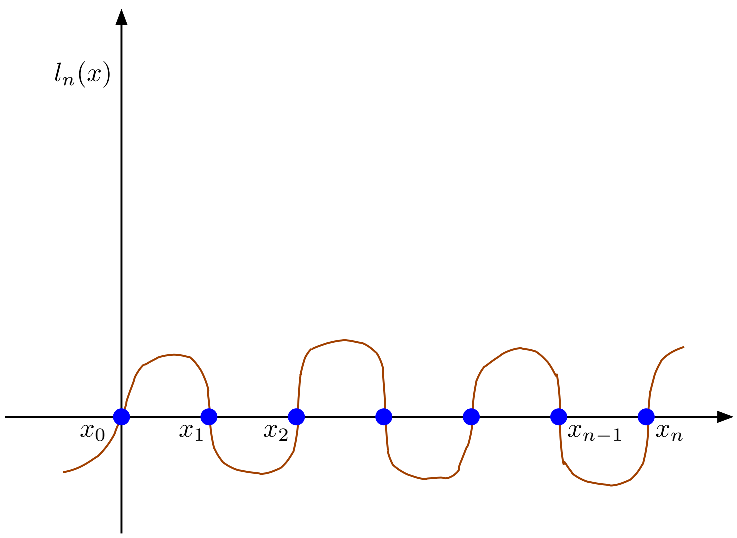
Using Lagrange polynomials $\mathcal{P}_n(x)$ $(= x)$ is written as

$$f(x) = \sum_{i=0}^{n} y_i l_i(x)$$

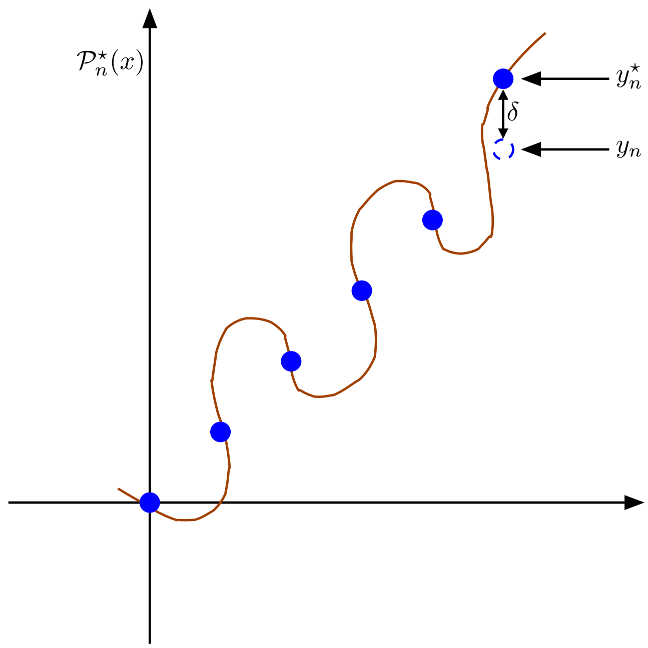
Let us "shift" y_n by a small amount δ. The new value is $y_n^\star = y_n + \delta$. The updated interpolant $\mathcal{P}_n^\star(x)$ then becomes:

$$\mathcal{P}_n^\star(x) = \sum_{i=0}^{n-1} y_i l_i(x) + y_n^\star l_n(x)$$

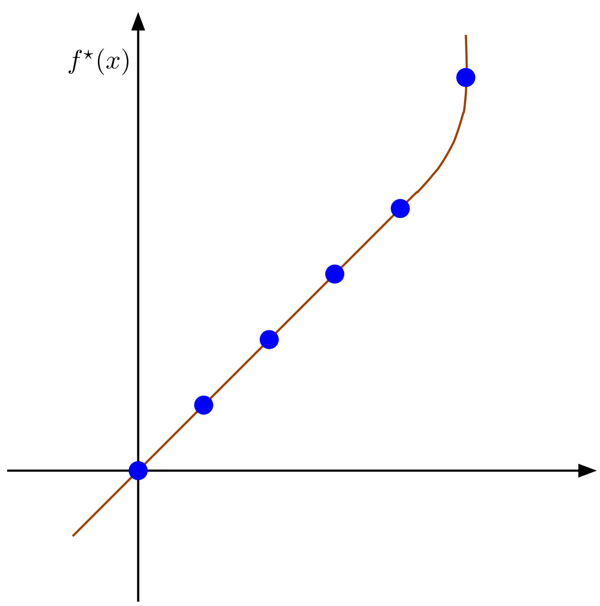
Thus, $\mathcal{P}_n^\star(x) - \mathcal{P}_n(x) = \delta \cdot l_n(x)$. Note that l_n is a function that "oscillates" through zero several times:



Thus, $\mathcal{P}_n^\star(x)$ looks like



What we observe is that a *local* change in y-values caused a *global* (and drastic) change in $\mathcal{P}_n(x)$. Perhaps the "real" function f would have exhibited a more graceful and localized change, e.g.:



We will use the following theorem to compare the "real" function f being sampled, and the reconstructed interpolant $\mathcal{P}_n(x)$.

Theorem 1. *Let*

- $x_0 < x_1 < \ldots < x_{n-1} < x_n$
- $y_n = f(x_n)$, $k = 0, 1, \ldots, n$, *where f is a function which is n-times differentiable with continuous derivatives*
- $\mathcal{P}_n(x)$ *is a polynomial that interpolates* $(x_0, y_0), (x_1, y_1) \ldots, (x_n, y_n)$

then for any $x \in (x_0, x_n)$, there exists a $\theta = \theta(x) \in (x_0, x_n)$ such that

$$f(x) - \mathcal{P}_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)(x - x_1) \ldots (x - x_n)$$

This theorem may be difficult to apply directly since:

- θ is not known
- θ changes with x
- The $(n+1)$-th derivative $f^{(n+1)}(x)$ may not be fully known.

However, we can use it to derive a conservative bound:

Theorem 2. *If $M = \max_{x \in [x_0, x_n]} |f^{(n+1)}(x)|$ and $h = \max_{0 \le i \le n} |x_{i+1} - x_i|$, then*

$$|f(x) - \mathcal{P}_n(x)| \le \frac{M h^{n+1}}{4(n+1)}$$

for all $x \in [x_0, x_n]$.

How good is this, especially when we keep adding more and more data points (e.g., $n \to \infty$ and $h \to 0$), really depends on the higher order derivatives of $f(x)$. For example, $f(x) = \sin(x)$, $x \in [0, 2\pi]$, alll derivatives of f are $\pm \sin(x)$ or $\pm \cos(x)$. Thus, $|f^{(k)}(x)| \le 1$ for *any* k. In this case, $M = 1$, and as we add more (and denser) data points, we have

$$|f(x) - \mathcal{P}_n(x)| \le \frac{M h^{n+1}}{4(n+1)} \xrightarrow[h \to 0]{n \to \infty} 0$$

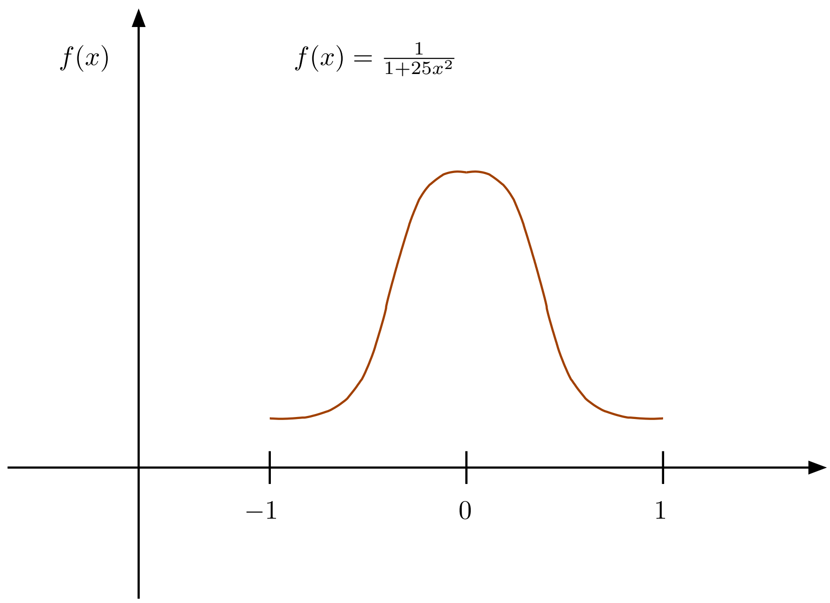
For some functions, however, the values of $|f^{(k)}(x)|$ grow vastly as $k \to \infty$ (i.e., when we introduce additional points). For example,

$$f(x) = \frac{1}{x}, \quad x \in (0.5, 1) \Rightarrow |f^{(n)}(x)| = n! \frac{1}{x^{n+1}}, M = \max_{x \in (0.5,1)} |f^{(n)}(x)| = n! 2^{n+1}$$

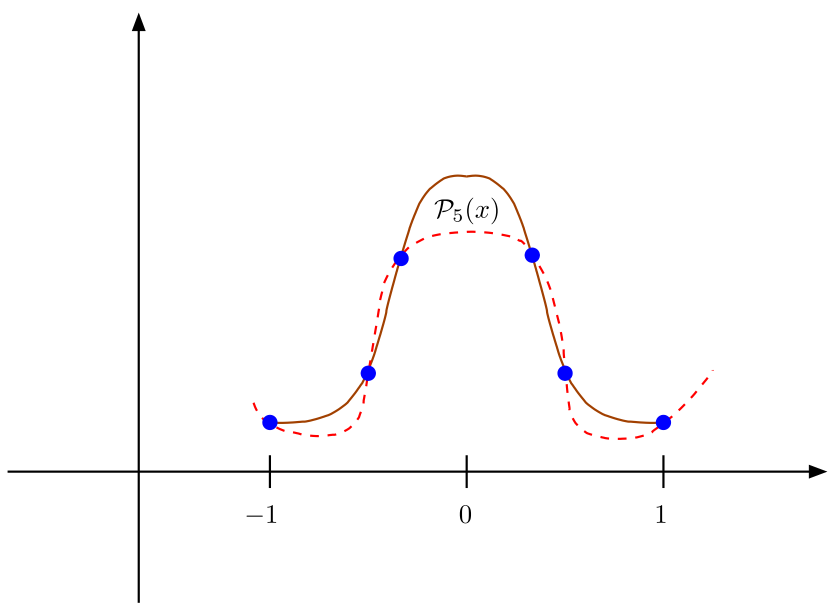
In this case, as $n \to \infty$:

$$\frac{M h^n}{4n} = \frac{n! 2^{n+1} h^n}{4n} \xrightarrow{n \to \infty} \infty$$

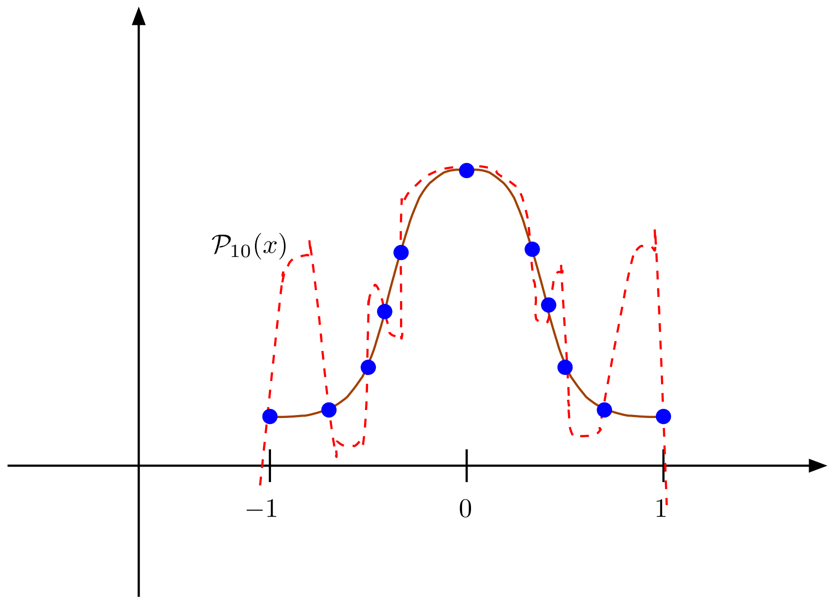
Another commonly cited example is *Runge's function*:



Approximation with a degree-5 polynomial:



Approximation with a degree-10 polynomial:



Thus, in this case, the polynomial $\mathcal{P}_k(x)$ do *not* uniformly converge to $f(x)$ as we add more points.

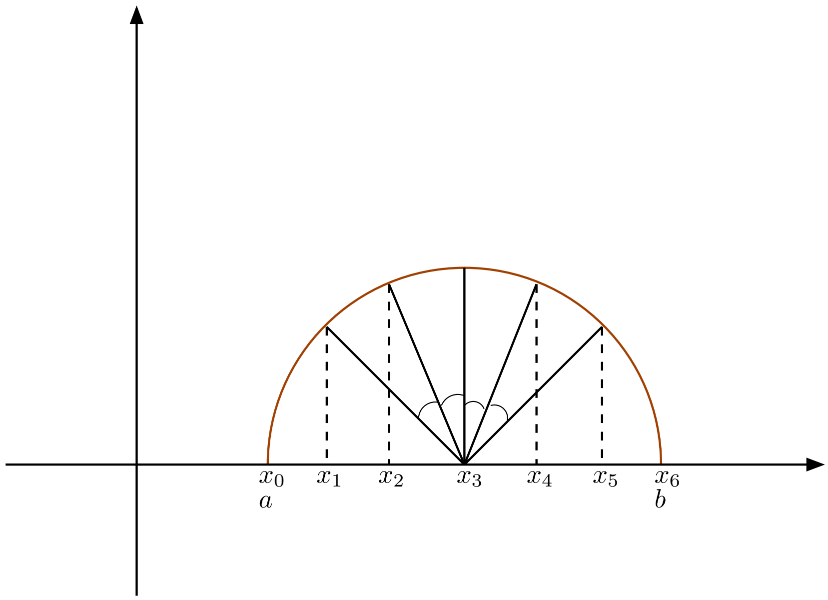
A possible improvement stems from the following idea:

$$f(x) - \mathcal{P}_n(x) = \underbrace{\frac{f^{(n+1)}(\theta)}{(n+1)!}}_{\text{this can be arbitrary}} \underbrace{(x-x_0)\ldots(x-x_n)}_{\text{select points to minimize this product}}$$

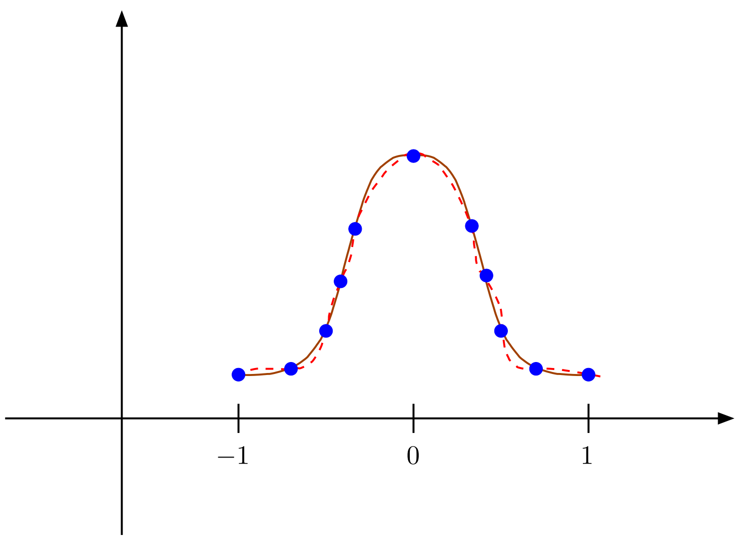
The value of the product $(x-x_0)\ldots(x-x_n)$ is minimized by selecting the x_i's as the *Chebyshev points*. If the interpolation interval is $[a, b]$, the Chebyshev points are given by:

$$x_i = \frac{a+b}{2} + \frac{a-b}{2}\cos\left(\frac{i\pi}{n}\right), \quad i = 0, 1, 2, \ldots, n$$

Graphically, these points are the projections on the x-axis of the $n+1$ points located along the half circle with diameter the interval $[a, b]$ at equal arc-lengths:



Now, we can re-try Runge's function using Chebyshev points:



In fact, it is possible to show that using Chebyshev points, we can guarantee that

$$|f(x) - \mathcal{P}_n(x)| \xrightarrow{n\to\infty} 0$$

provided that over $[a, b]$ both $f(x)$ and its derivative $f'(x)$ remain bounded (the benefit is that this condition does not place restrictions on higher-order derivatives of $f(x)$).