We saw three methods for polynomial interpolation (Vandermonde, Lagrange, Newton). It is important to understand that all three methods compute (in theory) the same exact interpolant $P_n(x)$, just following different paths which may be better or worse from a computational perspective. The question however remains:

- How accurate is this interpolation, or in other words,
- How close is $P_n(x)$ to the “real” function $f(x)$?

Example:

\[
\begin{align*}
(x_0, y_0) &= (0, 0) \\
(x_1, y_1) &= (1, 1) \\
&\vdots \\
(x_n, y_n) &= (n, n)
\end{align*}
\]
Using Lagrange polynomials $P_n(x) (= x)$ is written as

$$f(x) = \sum_{i=0}^{n} y_i l_i(x)$$

Let us “shift” $y_n$ by a small amount $\delta$. The new value is $y^*_n = y_n + \delta$. The updated interpolant $P^*_n(x)$ then becomes:

$$P^*_n(x) = \sum_{i=0}^{n-1} y_i l_i(x) + y^*_n l_n(x)$$

Thus, $P^*_n(x) - P_n(x) = \delta \cdot l_n(x)$. Note that $l_n$ is a function that “oscillates” through zero several times:
Thus, $\mathcal{P}_n^*(x)$ looks like

![Graph of $\mathcal{P}_n^*(x)$](image1)

What we observe is that a local change in $y$-values caused a global (and drastic) change in $\mathcal{P}_n(x)$. Perhaps the “real” function $f$ would have exhibited a more graceful and localized change, e.g.:

![Graph of $f^*(x)$](image2)
We will use the following theorem to compare the “real” function \( f \) being sampled, and the reconstructed interpolant \( P_n(x) \).

**Theorem 1.** Let

- \( x_0 < x_1 < \ldots < x_{n-1} < x_n \)
- \( y_n = f(x_n), \ k = 0, 1, \ldots, n \), where \( f \) is a function which is \( n \)-times differentiable with continuous derivatives
- \( P_n(x) \) is a polynomial that interpolates \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\)

then for any \( x \in (x_0, x_n) \), there exists a \( \theta = \theta(x) \in (x_0, x_n) \) such that

\[
|f(x) - P_n(x)| = \sum_{k=1}^{n+1} \frac{f^{(k)}(\theta)}{(k)!} (x - x_0)(x - x_1) \ldots (x - x_n)
\]

This theorem may be difficult to apply directly since:

- \( \theta \) is not known
- \( \theta \) changes with \( x \)
- The \((n + 1)\)-th derivative \( f^{(n+1)}(x) \) may not be fully known.

However, we can use it to derive a conservative bound:

**Theorem 2.** If \( M = \max_{x \in [x_0, x_n]} |f^{(n+1)}(x)| \) and \( h = \max_{0 \leq i \leq n} |x_{i+1} - x_i| \), then

\[
|f(x) - P_n(x)| \leq \frac{Mh^{n+1}}{4(n + 1)}
\]

for all \( x \in [x_0, x_n] \).

How good is this, especially when we keep adding more and more data points (e.g., \( n \to \infty \) and \( h \to 0 \)), really depends on the higher order derivatives of \( f(x) \). For example, \( f(x) = \sin(x), x \in [0, 2\pi] \), all derivatives of \( f \) are \( \pm \sin(x) \) or \( \pm \cos(x) \). Thus, \( |f^{(k)}(x)| \leq 1 \) for any \( k \). In this case, \( M = 1 \), and as we add more (and denser) data points, we have

\[
|f(x) - P_n(x)| \leq \frac{Mh^{n+1}}{4(n + 1)} \quad \text{as} \quad h \to 0
\]

For some functions, however, the values of \( |f^{(k)}(x)| \) grow vastly as \( k \to \infty \) (i.e., when we introduce additional points). For example,

\[
f(x) = \frac{1}{x}, \quad x \in (0.5, 1) \Rightarrow |f^{(n)}(x)| = n! \frac{1}{x^{n+1}}, M = \max_{x \in (0.5, 1)} |f^{(n)}(x)| = n!2^{n+1}
\]

In this case, as \( n \to \infty \):

\[
\frac{Mh^n}{4n} = \frac{n!2^{n+1}h^n}{4n} \quad \text{as} \quad n \to \infty
\]

Another commonly cited example is Runge’s function:
Approximation with a degree-5 polynomial:
Approximation with a degree-10 polynomial:

\[ P_{10}(x) - 101 \]

Thus, in this case, the polynomial \( P_k(x) \) do not uniformly converge to \( f(x) \) as we add more points.

A possible improvement stems from the following idea:

\[
f(x) - P_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!} (x - x_0) \cdots (x - x_n)
\]

this can be arbitrary

select points to minimize this product

The value of the product \((x - x_0) \cdots (x - x_n)\) is minimized by selecting the \(x_i\)'s as the Chebyshev points. If the interpolation interval is \([a, b]\), the Chebyshev points are given by:

\[
x_i = \frac{a + b}{2} + \frac{a - b}{2} \cos \left( \frac{i\pi}{n} \right), \quad i = 0, 1, 2, \ldots, n
\]

Graphically, these points are the projections on the \(x\)-axis of the \(n + 1\) points located along the half-circle with diameter the interval \([a, b]\) at equal arc-lengths:
Now, we can re-try Runge’s function using Chebyshev points:

In fact, it is possible to show that using Chebyshev points, we can guarantee that

\[ |f(x) - P_n(x)| \xrightarrow{n \to \infty} 0 \]

provided that over \([a, b]\) both \(f(x)\) and its derivative \(f'(x)\) remain bounded (the benefit is that this condition does not place restrictions on higher-order derivatives of \(f(x)\)).