CS412: Lecture #12

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Although using Chebyshev points mitigates some of the drawbacks of high-order polynomial interpolants, this is still a non-ideal solution, since:

- We do not always have the flexibility to pick the x_i's.
- Polynomial interpolants of high degree typically require more than $O(n)$ computational cost to construct.
- Local changes in the data points affect the entire extent of the interpolant.

Piecewise Polynomials

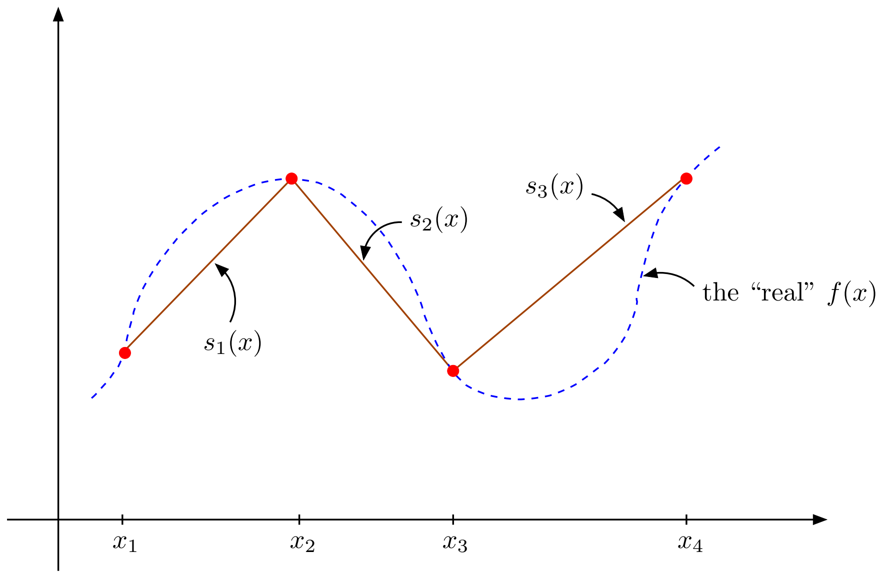
A better remedy is to use piecewise polynomials. Assume that the x-values $\{x_i\}_{i=1}^n$ are sorted in ascending order:

$$a = x_1 < x_2 < \ldots < x_n = b$$

Define $I_k = [x_k, x_{k+1}]$ and $h_k = |x_{k+1} - x_k|$. We also define the polynomials $s_1(x), s_2(x), \ldots, s_{n-1}(x)$ and use each of them to define the interpolant $s(x)$ at the respective interval I_k:

$$s(x) = \begin{cases} s_1(x), x \in I_1 \\ s_2(x), x \in I_2 \\ \vdots \\ s_{n-1}(x), x \in I_{n-1} \end{cases}$$

The benefit of using piecewise polynomial interpolants is that each $s_k(x)$ can be relatively low order and thus, non-oscillatory and easier to compute. The simplest piecewise polynomial interpolant is a *piecewise linear* curve:



In this case, every s_k can be written out explicitly as:

$$s_k(x) = y_k + \frac{y_{k+1} - y_k}{x_{k+1} - x_k}(x - x_k)$$

The next step is to examine the error $e(x) = f(x) - s_k(x)$ in the interval I_k. From the theorem we presented in the last lecture, we have that, for any $x \in I_k$ there is a $\theta_k = \theta(x_k) \in I_k$ such that:

$$e(x) = f(x) - s_k(x) = \frac{f''(\theta)}{2} \underbrace{(x - x_k)(x - x_{k+1})}_{q(x)} \tag{1}$$

We are interested in the *maximum* value of $|q(x)|$ in order to determine a bound for the error. $q(x)$ is a quadratic function which crosses zero at x_k and x_{k+1}, thus the extreme value is obtained at the midpoint $x_m = (x_k + x_{k+1})/2$. Thus,

$$|q(x)| \leq |q(x_m)| = \left(\frac{x_{k+1} - x_k}{2}\right)^2 = \frac{h_k^2}{4}$$

for all $x \in I_k$. Using equation (1) gives:

$$\begin{aligned} |f(x) - s_k(x)| &\leq \max_{x \in I_k} \left|\frac{f''(x)}{2}\right| \cdot \max_{x \in I_k} |(x - x_k)(x - x_{k+1})| \\ &= \max_{x \in I_k} \left|\frac{f''(x)}{2}\right| \cdot \frac{h_k^2}{4} \\ \Rightarrow |f(x) - s_k(x)| &\leq \frac{1}{8} \max_{x \in I_k} |f''(x)| \cdot h_k^2 \end{aligned}$$

for all $x \in I_k$.

Additionally, if we assume all data points are equally spaced, i.e.,

$$h_1 = h_2 = \ldots = h_{n-1} = h = \left(\frac{b-a}{n-1}\right)$$

we can additionally write:

$$|f(x) - s(x)| \leq \frac{1}{8} \max_{x\in[a,b]} |f''(x)| \cdot h^2$$

We often express the quantity on the right hand side using the "infinity norm" of a given function, defined as

$$||f||_\infty = \max_{x\in[a,b]} |f(x)|$$

Thus, using this notation:

$$\boxed{|f(x) - s(x)| \leq \frac{1}{8}||f''||_\infty \cdot h^2}$$

Note that

- As $h \to 0$, the maximum discrepancy between f and s vanishes (proportionally to h^2)
- As we introduce more points, the quality of the approximation increases consistently, since the criterion above only considers the second derivative $f''(x)$ and not any higher order.

Piecewise cubic interpolation

In this approach, each $s_k(x)$ is a *cubic* polynomial, designed such that it interpolates the four data points:

$$(x_{k-1}, y_{k-1}), (x_k, y_k), (x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2})$$

As we will see, the benefit is that the error can be made even smaller than with piecewise linear curves; the drawback is that $s(x)$ can develop "kinks" (or corners) where two pieces s_k and s_{k+1} are joined.

Error of piecewise cubics:

$$f(x) - s_k(x) = \frac{f''''(\theta_k)}{4!} \underbrace{(x - x_{k-1})(x - x_k)(x - x_{k+1})(x - x_{k+2})}_{q(x)}$$

An analysis similar to the linear case can show that

$$|q(x)| \leq \frac{9}{16} \max\{h_{k-1}, h_k, h_{k+1}\}^4$$

If we again assume that $h_1 = h_2 = \ldots = h_{n-1} = h$, the error bound becomes:

$$\begin{aligned} |f(x) - s(x)| &\leq \frac{1}{24}||f''''||_\infty \frac{9}{16}h^4 \\ \Rightarrow |f(x) - s(x)| &\leq \frac{9}{384}||f''''||_\infty h^4 \end{aligned}$$

The next possibility we shall consider, is a piecewise cubic curve

$$s(x) = \begin{cases} s_1(x), x \in I_1 \\ s_2(x), x \in I_2 \\ \vdots \\ s_{n-1}(x), x \in I_{n-1} \end{cases}$$

where each $s_k(x) = a_3^{(k)}x^3 + a_2^{(k)}x^2 + a_1^{(k)}x + a_0^{(k)}$ and the coefficients $a_i^{(j)}$ are chosen as to *force* that the curve has continuous values, first and second derivatives:

$$\begin{aligned} s_k(x_{k+1}) &= s_{k+1}(x_{k+1}) \\ s'_k(x_{k+1}) &= s'_{k+1}(x_{k+1}) \\ s''_k(x_{k+1}) &= s''_{k+1}(x_{k+1}) \end{aligned}$$

The curve constructed this way is called a *cubic spline* interpolant.