Numerical Integration

For the rectangle rule, we have the following local error:

\[ e_i = \left| \int_{x_i}^{x_{i+1}} f(x)dx - \int_{x_i}^{x_{i+1}} f(x)dx \right| = \left| \int_{x_i}^{x_{i+1}} [f(x) - f(x)]dx \right| \quad (1) \]

We shall seek to obtain an upper bound for the integral in equation (1). Let us remember Taylor’s formula, applied to \( f(x) \) in the vicinity of \( x_i \):

\[ f(x) = f(x_i) + f'(c_i)(x - x_i), \quad \text{where } c_i \in (x_i, x_{i+1}) \]

Thus,

\[ e_i = \left| - \int_{x_i}^{x_{i+1}} f'(c_i)(x - x_i) \right| \leq \int_{x_i}^{x_{i+1}} |f'(c_i)||x - x_i|dx \]

\[ \Rightarrow e_i \leq \int_{x_i}^{x_{i+1}} ||f'||_{\infty} |x - x_i| dx \leq ||f'||_{\infty} \int_{x_i}^{x_{i+1}} (x - x_i)dx \]

\[ \Rightarrow e_i \leq ||f'||_{\infty} \left( \frac{(x - x_i)^2}{2} \right)_{x_i}^{x_{i+1}} = \frac{1}{2} ||f'||_{\infty} (x_{i+1} - x_i)^2 \]

\[ \Rightarrow e_i \leq \frac{1}{2} ||f'||_{\infty} \cdot h_i^2 \]

The global error is defined as:

\[ e = |I_{\text{rule}} - I_{\text{analytic}}| = \sum_{i=0}^{n-1} |I_i,\text{rule} - I_i,\text{analytic}| \leq \sum_{i=0}^{n-1} |I_i,\text{rule} - I_i,\text{analytic}| = \sum_{i=0}^{n-1} e_i \]

For example, if \( h = \text{constant} \), for the rectangle rule we have:
\[ e \leq \sum_{i=0}^{n-1} e_i = n \cdot \frac{1}{2} \| f' \|_\infty \cdot h^2 \Rightarrow e_{global} \leq \frac{b - a}{2} \| f' \|_\infty h \]

What we observe is that, for the rectangle rule:

- Local error = \( O(h^2) \)
- Global error = \( O(h) \)

In general, we always get that if the local error is \( O(h^{d+1}) \) the global will be \( O(h^d) \); additionally, in this case the numerical integration rule is called \( d \)-order accurate (e.g., rectangle rule is 1st order accurate).

**Midpoint Rule**

Here, we approximate \( f(x) \approx f \left( \frac{x_k + x_{k+1}}{2} \right) \). Thus, the integral becomes

\[ I = \int_a^b f(x)dx \approx (b - a) f \left( \frac{a + b}{2} \right) \]

Composite rule, assuming \( h = \text{constant} \)
\[
I = \int_a^b f(x)dx \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i) f \left( \frac{x_i + x_{i+1}}{2} \right) = \frac{b-a}{n} \sum_{i=0}^{n-1} f \left( \frac{x_i + x_{i+1}}{2} \right)
\]

**Local error analysis**

We use the (2nd order) Taylor’s formula around the point \( x_m = (x_i + x_{i+1})/2 \).

\[
f(x) = f(x_m) + f'(x_m)(x - x_m) + \frac{f''(c_i)}{2}(x - x_m)^2 \quad \text{where} \quad c_i \in (x_i, x_{i+1})
\]

\[
e_i = \left| \int_{x_i}^{x_{i+1}} [f(x_m) - f(x)]dx \right| = \left| f(x_m) \int_{x_i}^{x_{i+1}} (x - x_m)dx + \int_{x_i}^{x_{i+1}} \frac{f''(c_i)}{2}(x - x_m)^2 \right|
\]

Note that \( \int_{x_i}^{x_{i+1}} (x - x_m)dx = \left| \frac{(x-x_m)^2}{2} \right|_{x_i}^{x_{i+1}} = \frac{h_i^2}{2} - \frac{h_i^2}{2} = 0 \). Thus,

\[
e_i \leq \frac{1}{2} \|f''\|_\infty \int_{x_i}^{x_{i+1}} (x - x_m)^2 dx = \frac{1}{2} \|f''\|_\infty \left| \frac{(x-x_m)^3}{3} \right|_{x_i}^{x_{i+1}}
\]

\[
\Rightarrow e_i \leq \frac{1}{24} \|f''\|_\infty h_i^3
\]

**Global error**

\[
e_{\text{global}} \leq \sum_{i=0}^{n-1} e_i \Rightarrow e_{\text{global}} \leq \frac{b-a}{24} \|f''\|_\infty h^2
\]

Thus, the midpoint rule is **2nd order** accurate.
Trapezoidal rule

In this case, $f$ is approximated in $[a, b]$ with the straight line drawn between $(a, f(a))$ and $(b, f(b))$. 

\[
I = \int_a^b f(x)dx \approx (b - a) \frac{f(a) + f(b)}{2} = I_{\text{trap}}
\]

To generate the corresponding composite rule, we write:

\[
I_i = \int_{x_i}^{x_{i+1}} f(x)dx \approx (x_{i+1} - x_i) \frac{f(x_i) + f(x_{i+1})}{2} = h_i \frac{f(x_i) + f(x_{i+1})}{2}
\]

Assuming $h = \text{constant}$, this gives:

\[
I = \sum_{i=0}^{n-1} I_i \approx \sum_{i=0}^{n-1} h \frac{f(x_i) + f(x_{i+1})}{2}
\]

\[
= \frac{b - a}{2n} (f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n))
\]
Note that due to the simple formula for the trapezoidal area, we did not have to write the approximating polynomial

\[ p(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \]

explicitly. Also, the result of integrating \(\int_a^b p(x)\,dx\) results in a very simple formula \([b - a]\frac{f(a) + f(b)}{2}\), even “simpler” than the formula for \(p\) itself!

**Local error analysis**

Estimating the local error can be somewhat delicate with the trapezoidal rule. We will, in this case, use a formula from the theory of interpolating polynomials we saw before:

**Theorem 1.** If \(P(x)\) is an \(n\)-degree polynomial interpolating \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\), then for every \(x \in [x_0, x_n]\) we have,

\[ f(x) - P(x) = \frac{f^{(n+1)}(c)}{(n + 1)!} (x - x_0)(x - x_1) \ldots (x - x_n) \]

Caution: \(c\) is not a constant! It depends on the particular \(x\) we choose in this theorem.

For the trapezoidal rule, we effectively use a linear \((n = 1)\) interpolant. When \(x \in [x_i, x_{i+1}]\), we have:

\[ f(x) - P_i(x) = \frac{f''(c)}{2} (x - x_i)(x - x_{i+1}) \quad \text{where } c_i \in (x_i, x_{i+1}) \]

Thus,

\[
\begin{align*}
\varepsilon_i &= \left| \int_{x_i}^{x_{i+1}} [f(x) - p_i(x)]\,dx \right| = \left| \int_{x_i}^{x_{i+1}} \frac{f''(c_i)}{2} (x - x_i)(x - x_{i+1}) \,dx \right| \\
&\leq \int_{x_i}^{x_{i+1}} \frac{f''(c_i)}{2} \left| (x - x_i)(x - x_{i+1}) \right|\,dx \\
&\leq \frac{1}{2} \|f''\|_{\infty} \int_{x_i}^{x_{i+1}} \left| (x - x_i)(x - x_{i+1}) \right|\,dx
\end{align*}
\]

The only reason we can meaningfully continue at this point is to recognize that \((x-x_i)(x-x_{i+1}) \leq 0\) in \([x_i, x_{i+1}]\). Thus, \(|(x-x_i)(x-x_{i+1})| = -(x-x_i)(x-x_{i+1})\) and remove this way the absolute value in the integral above. This is not the case in general, where we won’t be able to remove the absolute value (see Simpson’s rule next). We can verify that
\[
\int_{x_i}^{x_{i+1}} |(x - x_i)(x - x_{i+1})| = -\int_{x_i}^{x_{i+1}} (x - x_i)(x - x_{i+1}) = \frac{h_i^3}{6}
\]

Putting everything together:

\[
e_i \leq \frac{1}{2} ||f''||_\infty h_i^3 \Rightarrow e_i \leq \frac{1}{12} ||f''||_\infty h_i^3
\]

For the global error:

\[
e \leq \sum_{i=0}^{n-1} e_i \leq \frac{n}{12} ||f''||_\infty h_i^3 \xrightarrow{nh=b-a} e \leq \frac{b-a}{12} ||f''||_\infty h^2
\]

Thus, trapezoidal rule is also 2nd order accurate.