CS412: Lecture #21

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Numerical Integration

For the rectangle rule, we have the following local error:

$$e_i = \left| \int_{x_i}^{x_{i+1}} f(x_k)dx - \int_{x_i}^{x_{i+1}} f(x)dx \right| = \left| \int_{x_i}^{x_{i+1}} [f(x_i) - f(x)]dx \right| \tag{1}$$

We shall seek to obtain an upper bound for the integral in equation (1). Let us remember Taylor's formula, applied to $f(x)$ in the vicinity of x_i:

$$f(x) = f(x_i) + f'(c_i)(x - x_i), \qquad \text{where } c_i \in (x_i, x_{i+1})$$

Thus,

$$\begin{aligned}
e_i &= \left| -\int_{x_i}^{x_{i+1}} f'(c_i)(x - x_i) \right| \leq \int_{x_i}^{x_{i+1}} |f'(c_i)||x - x_i|dx \\
\Rightarrow e_i &\leq \int_{x_i}^{x_{i+1}} ||f'||_\infty \underbrace{|x - x_i|}_{\geq 0} dx \leq ||f'||_\infty \int_{x_i}^{x_{i+1}} (x - x_i)dx \\
\Rightarrow e_i &\leq ||f'||_\infty \left[\frac{(x - x_i)^2}{2} \right]_{x_i}^{x_{i+1}} = \frac{1}{2}||f'||_\infty (x_{i+1} - x_i)^2 \\
\Rightarrow e_i &\leq \frac{1}{2}||f'||_\infty \cdot h_i^2
\end{aligned}$$

The global error is defined as:

$$e = |I_{\mathsf{rule}} - I_{\mathsf{analytic}}| = \left| \sum_{i=0}^{n-1} [I_{i,\mathsf{rule}} - I_{i,\mathsf{analytic}}] \right| \leq \sum_{i=0}^{n-1} |I_{i,\mathsf{rule}} - I_{i,\mathsf{analytic}}| = \sum_{i=0}^{n-1} e_i$$

For example, if $h = \mathsf{constant}$, for the rectangle rule we have:

$$e \leq \sum_{i=0}^{n-1} e_i = n \cdot \frac{1}{2} ||f'||_\infty \cdot h^2 \Rightarrow \boxed{e_{\textsf{global}} \leq \frac{b-a}{2} ||f'||_\infty h}$$

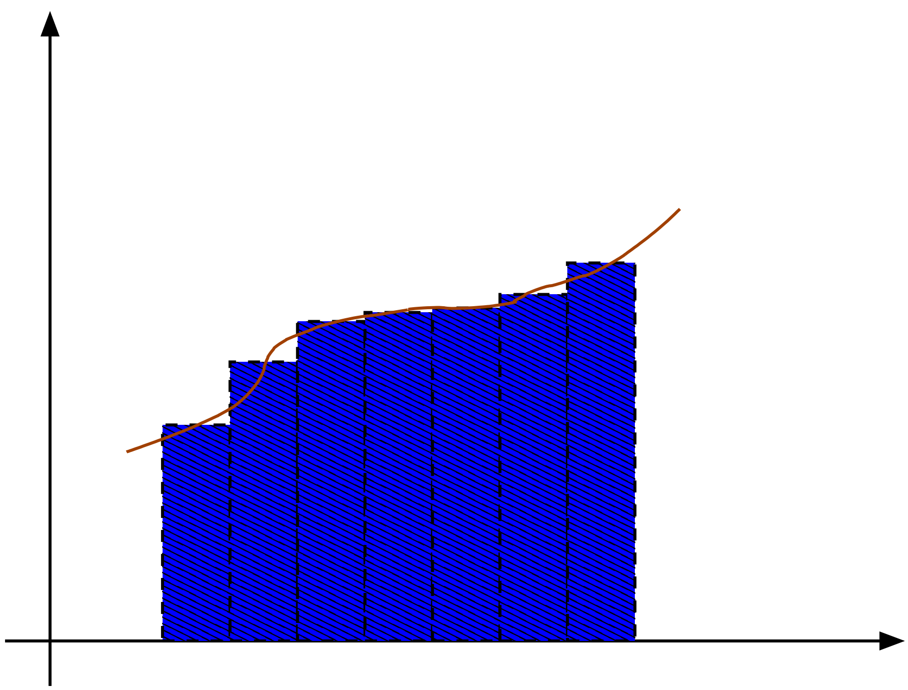
What we observe is that, for the rectangle rule:

- Local error $= O(h^2)$
- Global error $= O(h)$

In general, we always get that if the local error is $O(h^{d+1})$ the global will be $O(h^d)$; additionally, in this case the numerical integration rule is called d-order accurate (e.g., rectangle rule is 1st order accurate).

Midpoint Rule

Here, we approximate $f(x) \approx f\left(\frac{x_k + x_{k+1}}{2}\right)$. Thus, the integral becomes



$$I = \int_a^b f(x)dx \approx (b-a) f\left(\frac{a+b}{2}\right)$$

Composite rule, assuming $h = \textsf{constant}$

$$I = \int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \underbrace{(x_{i+1} - x_i)}_{=h=(b-a)/n} f\left(\frac{x_i + x_{i+1}}{2}\right) = \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)$$

Local error analysis

We use the (2nd order) Taylor's formula around the point $x_m = (x_i + x_{i+1})/2$.

$$f(x) = f(x_m) + f'(x_m)(x - x_m) + \frac{f''(c_i)}{2}(x - x_m)^2 \qquad \text{where } c_i \in (x_i, x_{i+1})$$

$$e_i = \left| \int_{x_i}^{x_{i+1}} [f(x_m) - f(x)dx] \right| = \left| f(x_m) \int_{x_i}^{x_{i+1}} (x - x_m)dx + \int_{x_i}^{x_{i+1}} \frac{f''(c_i)}{2}(x - x_m)^2 \right|$$

Note that $\int_{x_i}^{x_{i+1}} (x - x_m)dx = \left| \frac{(x-x_m)^2}{2} \right|_{x_i}^{x_{i+1}} = \frac{h_k^2}{8} - \frac{h_k^2}{8} = 0$. Thus,

$$\begin{aligned}
e_i &= \left| \frac{1}{2} \int_{x_i}^{x_{i+1}} f''(c_i)(x - x_m)^2 \right| \leq \frac{1}{2} \int_{x_i}^{x_{i+1}} |f''(c_i)| \, |x - x_m|^2 dx \\
&\leq \frac{1}{2} ||f''||_\infty \int_{x_i}^{x_{i+1}} (x - x_m)^2 dx = \frac{1}{2} ||f''||_\infty \left| \frac{(x - x_m)^3}{3} \right|_{x_i}^{x_{i+1}} \\
&\leq \frac{1}{2} ||f''||_\infty \left(\frac{h_i^3}{24} + \frac{h_i^3}{24} \right) \\
\Rightarrow e_i &\leq \frac{1}{24} ||f''||_\infty h_i^3
\end{aligned}$$

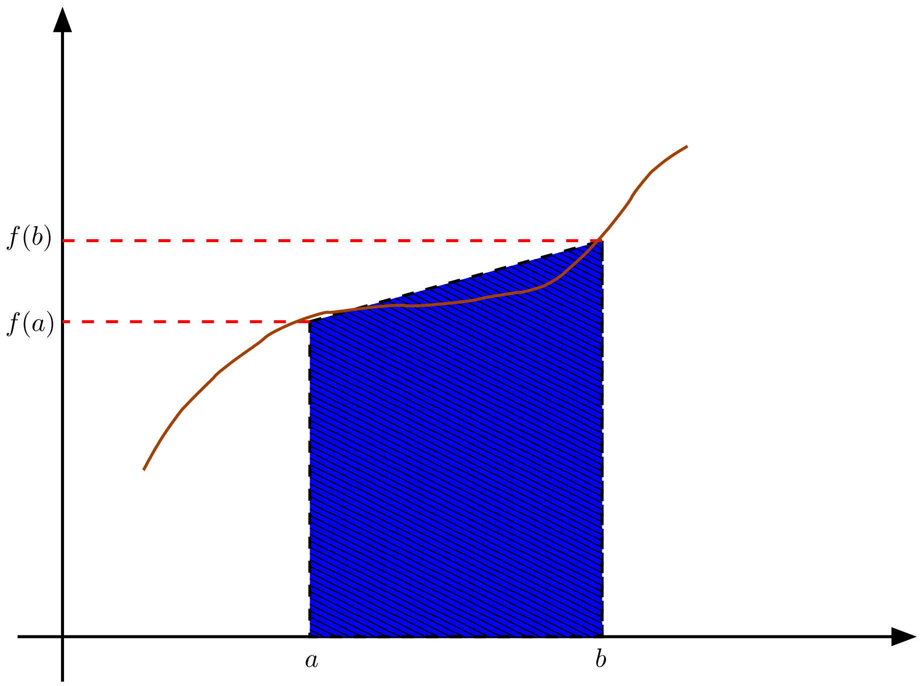
Global error

$$e_{\mathsf{global}} \leq \sum_{i=0}^{n-1} e_i \Rightarrow \boxed{e_{\mathsf{global}} \leq \frac{b-a}{24} ||f''||_\infty h^2}$$

Thus, the midpoint rule is 2*nd order* accurate.

Trapezoidal rule

In this case, f is approximated in $[a, b]$ with the straight line drawn between $(a, f(a))$ and $(b, f(b))$.



In this case,

$$I = \int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2} = I_{\mathsf{trap}}$$

To generate the corresponding composite rule, we write:

$$I_i = \int_{x_i}^{x_{i+1}} f(x)dx \approx (x_{i+1}-x_i)\frac{f(x_i)+f(x_{i+1})}{2} = h_i\frac{f(x_i)+f(x_{i+1})}{2}$$

Assuming $h = \mathsf{constant}$, this gives:

$$\begin{aligned} I &= \sum_{i=0}^{n-1} I_i \approx \sum_{i=0}^{n-1} h\frac{f(x_i)+f(x_{i+1})}{2} \\ &= \frac{b-a}{2n}\left(f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right) \end{aligned}$$

Note that due to the simple formula for the trapezoidal area, we did not have to write the approximating polynomial

$$p(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

explicitly. Also, the result of integrating $\int_a^b p(x)dx$ results in a very simple formula $\left[(b-a)\frac{f(a)+f(b)}{2}\right]$, even "simpler" than the formula for p itself!

Local error analysis

Estimating the local error can be somewhat delicate with the trapezoidal rule. We will, in this case, use a formula from the theory of interpolating polynomials we saw before:

Theorem 1. *If $\mathcal{P}(x)$ is an n-degree polynomial interpolating $(x_0, y_0), (x_1, y_1), \ldots,$ (x_n, y_n), then for every $x \in [x_0, x_n]$ we have,*

$$f(x) - \mathcal{P}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n)$$

Caution: c is *not* a constant! It depends on the particular x we choose in this theorem.

For the trapezoidal rule, we effectively use a linear ($n = 1$) interpolant. When $x \in [x_i, x_{i+1}]$, we have:

$$f(x) - \mathcal{P}_i(x) = \frac{f''(c_i)}{2}(x - x_i)(x - x_{i+1}) \qquad \text{where } c_i \in (x_i, x_{i+1})$$

Thus,

$$\begin{aligned}
e_i &= \left|\int_{x_i}^{x_{i+1}} [f(x) - p^i(x)]dx\right| = \left|\int_{x_i}^{x_{i+1}} \frac{f''(c_i)}{2}(x - x_i)(x - x_{i+1})\right| \\
&\leq \int_{x_i}^{x_{i+1}} \left|\frac{f''(c_i)}{2}\right| |(x - x_i)(x - x_{i+1})|dx \\
&\leq \frac{1}{2}||f''||_\infty \int_{x_i}^{x_{i+1}} |(x - x_i)(x - x_{i+1})|dx
\end{aligned}$$

The only reason we can meaningfully continue at this point is to recognize that $(x-x_i)(x-x_{i+1}) \leq 0$ in $[x_i, x_{i+1}]$. Thus, $|(x-x_i)(x-x_{i+1})| = -(x-x_i)(x-x_{i+1})$ and remove in this way the absolute value in the integral above. This is *not* the case in general, where we won't be able to remove the absolute value (see Simpson's rule next). We can verify that

$$\int_{x_i}^{x_{i+1}} |(x - x_i)(x - x_{i+1})| = -\int_{x_i}^{x_{i+1}} (x - x_i)(x - x_{i+1}) = \frac{h_i^3}{6}$$

Putting everything together:

$$e_i \leq \frac{1}{2}||f''||_\infty \frac{h_i^3}{6} \Rightarrow \boxed{e_i \leq \frac{1}{12}||f''||_\infty h_i^3}$$

For the global error:

$$e \leq \sum_{i=0}^{n-1} e_i \leq \frac{n}{12}||f''||_\infty h_i^3 \xrightarrow{nh=b-a} \boxed{e \leq \frac{b-a}{12}||f''||_\infty h^2}$$

Thus, trapezoidal rule is also 2nd order accurate.