# CS412: Lecture #23

Mridul Aanjaneya

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## Initial value problems for 1st order ordinary differential equations

In this last part of our class, we will turn our attention to differential equation problems, of the form: find the function $y(t) : [t_0, \infty) \rightarrow \mathbb{R}$ that satisfies the *ordinary differential equation* (ODE):

$$y'(t) = f(t, y(t)) \qquad \text{for a certain function } f \tag{1}$$

and $y(t_0) = y_0$. This is called an *initial value problem* (IVP), because the value of $y$ for $t > t_0$ are completely determined from the initial value $y_0$ and equation (1).

**Example #1:** The velocity of a vehicle over the time interval $[0, 5]$ satisfies $v(t) = t(t + 1)$. At time $t = 0$, the vehicle starts from position $x(0) = 5$. What is $x(t)$, $t \in [0, 5]$?

**Answer:** The problem is given by the IVP:

$$x'(t) = t(t + 1), \qquad x(0) = 5 \qquad\qquad \text{since } v(t) = x'(t)$$

**Example #2:** The concentration $y(t)$ of a chemical species is given by:

$$y'(t) = y(t)(t^2 + 1), \qquad y(0) = 1 \qquad\qquad \text{here } f(t, y) = y(t)(t^2 + 1)$$

Of course, in certain cases we can solve this differential equation *exactly*, for e.g., in the last example:

$$y'(t) = y(t)(t^2 + 1) \Rightarrow \frac{y'}{y} = t^2 + 1$$

Integrating both sides gives

$$\int_{t_0=0}^{t} \frac{y'}{y} d\tau = \int_{t_0=0}^{t} (\tau^2+1) d\tau \quad \Rightarrow \quad |\ln y|_{t_0=0}^{t} = \left| \frac{\tau^3}{3} + \tau \right|_0^t$$

$$\Rightarrow \ln y(t) - \underbrace{\ln y(0)}_{=0} = \frac{t^3}{3} + t \Rightarrow y(t) = e^{\frac{t^3}{3}+t}$$

However, we do *not* want to depend on our ability to solve the ODE exactly, since:

- An exact solution may not be analytically expressible in closed form.
- The exact solution may be too complicated and,
- (more importantly) the function $f(t,y)$ may not be available as a formula; e.g., it could result from a black box computer program.

**Solution:** *Approximate* the solution to the differential equation. General methodology ("1-step methods"):

- Consider *discrete* points in time

  $$t_0 < t_1 < t_2 < \ldots < t_n < \ldots$$

  If we set $\Delta t_k = t_{k+1} - t_k$ *and* $\Delta t_k = \Delta t = \mathsf{constant}$, then $t_k = t_0 + k\Delta t$.
- Use the notation $y_k = y(t_k)$.
- Use the values $t_k, y_k$ and the ODE $y'(t) = f(t,y)$ to approximate $y_{k+1}$.

**Method:**

$$\begin{aligned} y'(t) &= f(t,y) \\ \Rightarrow \int_{t_k}^{t_{k+1}} y'(\tau) d\tau &= \int_{t_k}^{t_{k+1}} f(\tau, y) d\tau \\ \Rightarrow \underbrace{y(t_{k+1})}_{=y_{k+1}} - \underbrace{y(t_k)}_{=y_k} &= \underbrace{\int_{t_k}^{t_{k+1}} f(\tau, y) d\tau}_{\text{Approximate using an integration rule!}} \end{aligned}$$

Thus,

$$\left.\begin{matrix} y_0 \\ t_0 \end{matrix}\right\} \to \left.\begin{matrix} y_1 \\ t_1 \end{matrix}\right\} \to \left.\begin{matrix} y_2 \\ t_2 \end{matrix}\right\} \to \left.\begin{matrix} y_3 \\ t_3 \end{matrix}\right\} \to \ldots$$

For example, approximating this integral with the *rectangle rule* $\int_a^b f(x)dx \approx f(a)(b-a)$ gives

$$y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y)d\tau \approx f(t_k, y_k)(t_{k+1} - t_k) \Rightarrow \boxed{y_{k+1} = y_k + \Delta t f(t_k, y_k)}$$

This method is called the *Forward Euler method*, or *Euler's method*, or *Explicit Euler's method*. It is easy to evaluate, plug in $t_k$, $y_k$ and obtain $y_{k+1}$.

Now, if we had used the "right-sided" rectangle rule $\int_a^b f(x)dx \approx f(b)(b-a)$, we would obtain:

$$y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y)d\tau \approx f(t_{k+1}, y_{k+1})\Delta t \Rightarrow \boxed{y_{k+1} = y_k + \Delta t f(t_{k+1}, y_{k+1})}$$

This method is called the *Backward Euler method*, or *Implicit Euler method.*

*Note:* We need to solve a (possibly nonlinear) equation to obtain $y_{k+1}$ ($y_{k+1}$ is not isolated in this equation).

One more variant: *trapezoidal rule* $\int_a^b f(x)dx = \frac{f(a)+f(b)}{2}(b-a)$.

$$\begin{aligned} y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y)d\tau &\approx \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2}\Delta t \\ &\Rightarrow \boxed{y_{k+1} = y_k + \frac{\Delta t}{2}\{f(t_k, y_k) + f(t_{k+1}, y_{k+1})\}} \end{aligned}$$

**Example:** $y'(t) = -ty^2$ using trapezoidal rule

$$y_{k+1} = y_k + \frac{\Delta t}{2}\{-t_k y_k^2 - t_{k+1} y_{k+1}^2\}$$

Let: $t_k = 0.9$, $y_k = 1$, and $\Delta t = 0.1$

$$\begin{aligned} y_{k+1} &= 1 + 0.05\{-0.9 - 1 \cdot y_{k+1}^2\} \\ \Rightarrow 0.05y_{k+1}^2 + y_{k+1} + 1.045 = 0 &\Rightarrow \text{solve quadratic to get } y_{k+1} \end{aligned}$$

**Another example:**

$$\left.\begin{array}{l} y'(t) = -2y(t) \\ y(0) = 1 \end{array}\right\} \text{exact solution } y(t) = e^{-2t}$$

Using Forward Euler:

$$\begin{aligned} y_{k+1} &= y_k + \Delta t f(t_k, y_k) \\ &= y_k - 2\Delta t y_k = (1 - 2\Delta t) y_k \end{aligned}$$

Thus,

$$\begin{aligned} y_1 &= (1 - 2\Delta t) y_0 \\ y_2 &= (1 - 2\Delta t) y_1 = (1 - 2\Delta t)^2 y_0 \\ &\vdots \\ y_k &= (1 - 2\Delta t)^k y_0 \end{aligned}$$

How does this behave when $\Delta t \to 0$?

$$(1 - 2\Delta t)^k = \left[ \left( 1 + \frac{1}{-\frac{1}{2\Delta t}} \right)^{-\frac{1}{2\Delta t}} \right]^{-2k\Delta t}$$

Using $\lim \left(1 + \frac{1}{x}\right)^x = e$,

$$\Rightarrow \lim_{\Delta t \to 0} (1 - 2\Delta t)^k = e^{-2k\Delta t} = e^{-2t_k}$$

Thus, when $\Delta t \to 0$, $y_k \to e^{-2t_k}$ (compare with exact solution $y(t) = e^{-2t}$).