Initial value problems for 1st order ordinary differential equations

In this last part of our class, we will turn our attention to differential equation problems, of the form: find the function $y(t) : [t_0, \infty) \to \mathbb{R}$ that satisfies the ordinary differential equation (ODE):

$$y'(t) = f(t, y(t)) \quad \text{for a certain function } f \quad (1)$$

and $y(t_0) = y_0$. This is called an initial value problem (IVP), because the value of $y$ for $t > t_0$ are completely determined from the initial value $y_0$ and equation (1).

Example #1: The velocity of a vehicle over the time interval $[0, 5]$ satisfies $v(t) = t(t + 1)$. At time $t = 0$, the vehicle starts from position $x(0) = 5$. What is $x(t)$, $t \in [0, 5]$?

Answer: The problem is given by the IVP:

$$x'(t) = t(t + 1), \quad x(0) = 5 \quad \text{since } v(t) = x'(t)$$

Example #2: The concentration $y(t)$ of a chemical species is given by:

$$y'(t) = y(t)(t^2 + 1), \quad y(0) = 1 \quad \text{here } f(t, y) = y(t)(t^2 + 1)$$

Of course, in certain cases we can solve this differential equation exactly, for e.g., in the last example:

$$y'(t) = y(t)(t^2 + 1) \Rightarrow \frac{y'}{y} = t^2 + 1$$
Integrating both sides gives

\[
\int_{t_0}^{t} \frac{y'}{y} d\tau = \int_{t_0}^{t} (\tau^2 + 1) d\tau \quad \Rightarrow \quad |\ln y|_{t_0}^{t} = \left| \frac{\tau^3}{3} + \tau \right|_{0}^{t} \\
\Rightarrow \ln y(t) - \ln y(0) = \frac{t^3}{3} + t \Rightarrow y(t) = e^{\frac{t^3}{3} + t}
\]

However, we do not want to depend on our ability to solve the ODE exactly, since:

- An exact solution may not be analytically expressible in closed form.
- The exact solution may be too complicated and,
- (more importantly) the function \( f(t, y) \) may not be available as a formula; e.g., it could result from a black box computer program.

**Solution:** Approximate the solution to the differential equation. General methodology (“1-step methods”):

- Consider discrete points in time
  
  \[ t_0 < t_1 < t_2 < \ldots < t_n < \ldots \]
  
  If we set \( \Delta t_k = t_{k+1} - t_k \) and \( \Delta t_k = \Delta t = \text{constant} \), then \( t_k = t_0 + k\Delta t \).
- Use the notation \( y_k = y(t_k) \).
- Use the values \( t_k, y_k \) and the ODE \( y'(t) = f(t, y) \) to approximate \( y_{k+1} \).

**Method:**

\[
\begin{align*}
\Rightarrow \int_{t_k}^{t_{k+1}} y'(\tau) d\tau &= \int_{t_k}^{t_{k+1}} f(\tau, y)d\tau \\
\Rightarrow y(t_{k+1}) - y(t_k) &= \int_{t_k}^{t_{k+1}} f(\tau, y)d\tau \\
\text{Approximate using an integration rule!}
\end{align*}
\]

Thus,

\[
\begin{align*}
\{ \frac{y_0}{t_0} \} & \rightarrow \{ \frac{y_1}{t_1} \} \rightarrow \{ \frac{y_2}{t_2} \} \rightarrow \{ \frac{y_3}{t_3} \} \rightarrow \ldots
\end{align*}
\]
For example, approximating this integral with the rectangle rule $\int_{a}^{b} f(x)dx \approx f(a)(b - a)$ gives

\[ y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y) d\tau \approx f(t_k, y_k)(t_{k+1} - t_k) \Rightarrow y_{k+1} = y_k + \Delta t f(t_k, y_k) \]

This method is called the Forward Euler method, or Euler’s method, or Explicit Euler’s method. It is easy to evaluate, plug in $t_k$, $y_k$ and obtain $y_{k+1}$.

Now, if we had used the “right-sided” rectangle rule $\int_{a}^{b} f(x)dx \approx f(b)(b - a)$, we would obtain:

\[ y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y) d\tau \approx f(t_{k+1}, y_{k+1}) \Delta t \Rightarrow y_{k+1} = y_k + \Delta t f(t_{k+1}, y_{k+1}) \]

This method is called the Backward Euler method, or Implicit Euler method.

Note: We need to solve a (possibly nonlinear) equation to obtain $y_{k+1}$ ($y_{k+1}$ is not isolated in this equation).

One more variant: trapezoidal rule $\int_{a}^{b} f(x)dx = \frac{f(a)+f(b)}{2} (b - a)$.

\[ y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} f(\tau, y) d\tau \approx \frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2} \Delta t \Rightarrow y_{k+1} = y_k + \frac{\Delta t}{2} \{f(t_k, y_k) + f(t_{k+1}, y_{k+1})\} \]

Example: $y'(t) = -ty^2$ using trapezoidal rule

\[ y_{k+1} = y_k + \frac{\Delta t}{2} \{-t_k y_k^2 - t_{k+1} y_{k+1}^2\} \]

Let: $t_k = 0.9$, $y_k = 1$, and $\Delta t = 0.1$

\[ y_{k+1} = 1 + 0.05\{-0.9 - 1 \cdot y_{k+1}^2\} \Rightarrow 0.05y_{k+1}^2 + y_{k+1} + 1.045 = 0 \Rightarrow \text{solve quadratic to get } y_{k+1} \]

Another example:

\[
\begin{align*}
y'(t) &= -2y(t) \\
y(0) &= 1
\end{align*}
\]

\[
\begin{align*}
\text{exact solution } y(t) &= e^{-2t}
\end{align*}
\]
Using Forward Euler:

\[ y_{k+1} = y_k + \Delta t f(t_k, y_k) \]
\[ = y_k - 2\Delta t y_k = (1 - 2\Delta t)y_k \]

Thus,

\[ y_1 = (1 - 2\Delta t)y_0 \]
\[ y_2 = (1 - 2\Delta t)y_1 = (1 - 2\Delta t)^2y_0 \]
\[ \vdots \]
\[ y_k = (1 - 2\Delta t)^ky_0 \]

How does this behave when \( \Delta t \to 0 \)?

\[ (1 - 2\Delta t)^k = \left[ \left( 1 + \frac{1}{2\Delta t} \right)^{-\frac{1}{2\Delta t}} \right]^{-2k\Delta t} \]

Using \( \lim (1 + \frac{1}{x})^x = e \),

\[ \Rightarrow \lim_{\Delta t \to 0} (1 - 2\Delta t)^k = e^{-2k\Delta t} = e^{-2t_k} \]

Thus, when \( \Delta t \to 0 \), \( y_k \to e^{-2t_k} \) (compare with exact solution \( y(t) = e^{-2t} \)).