CS412: Lecture #7

Mridul Aanjaneya

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Secant Method

Let the exact solution be a, i.e., $f(a) = 0$. Define $e_k = x_k - a$ and $f_k = f(x_k)$. Then,

$$\begin{aligned} e_{k+1} = x_{k+1} - a &= x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k - a \\ &= \frac{(x_{k-1} - a) f_k - (x_k - a) f_{k-1}}{f_k - f_{k-1}} \\ &= \frac{e_{k-1} f_k - e_k f_{k-1}}{f_k - f_{k-1}} \\ &= \frac{e_{k-1} f(a + e_k) - e_k f(a + e_{k-1})}{f_k - f_{k-1}} \end{aligned}$$

Expanding $f(a + e_k)$ using Taylor's series gives

$$\begin{aligned} e_{k+1} &= \frac{e_{k-1}(e_k f'(a) + e_k^2 f''(a)/2 + O(e_k^3)) - e_k(e_{k-1} f'(a) + e_{k-1}^2 f''(a)/2 + O(e_{k-1}^3))}{e_k f'(a) + e_k^2 f''(a)/2 + O(e_k^3) - (e_{k-1} f'(a) + e_{k-1}^2 f''(a)/2 + O(e_{k-1}^3))} \\ &= \frac{e_{k-1} e_k f''(a)(e_k - e_{k-1})/2 + O(e_{k-1}^4)}{(e_k - e_{k-1})(f'(a) + (e_k + e_{k-1}) f''(a)/2 + O(e_{k-1}^2))} \\ &= \frac{e_{k-1} e_k f''(a)}{2 f'(a)} + O(e_{k-1}^3) \end{aligned} \tag{1}$$

We want to find α such that $|e_{k+1}| = C|e_k|^\alpha$. Since the first term in equation (1) dominates the error, we ignore the cubic term and solve

$$\left| \frac{e_{k-1} e_k f''(a)}{2 f'(a)} \right| = C|e_k|^\alpha \tag{2}$$

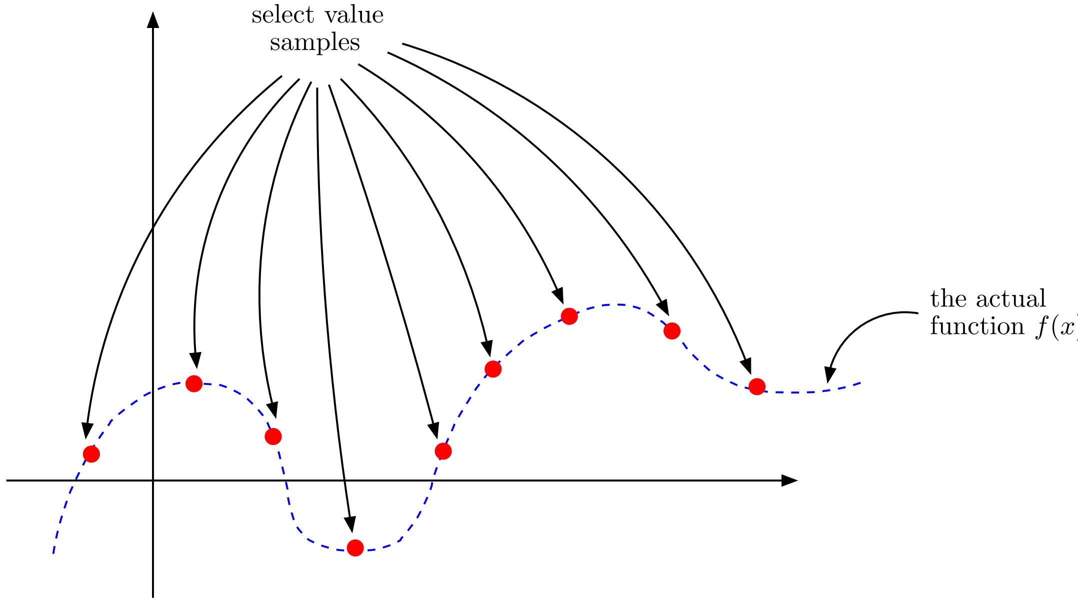
Canceling e_k from both sides and changing k to $k+1$ gives $|e_{k+1}|^{\alpha-1} = D|e_k|$, where $D = |f''(a)/(2Cf'(a))|$. Raising both sides by the power α gives

$$|e_{k+1}|^{\alpha(\alpha-1)} = D^\alpha |e_k^\alpha| \tag{3}$$

Equating equations (2) and (3) gives, $C = D^\alpha$ and $\alpha(\alpha - 1) = 1$. The negative solution can be discarded because we know that the order of convergence is positive. Thus, the order of convergence is $\alpha = (1 + \sqrt{5})/2 \approx 1.618$.

Interpolation

We are often interested in a certain function $f(x)$, but despite the fact that f may be defined over an entire interval of values $[a, b]$ (which may be the entire real line) we only know its precise value at select point $x_1, x_2, \ldots, x_N$.



There may be several good reasons why we could only have a limited number of values for $f(x)$, instead of its entire graph:

- Perhaps we do not have an analytic formula for $f(x)$ because it is the result of a complex process that is only observed experimentally. For example, $f(x)$ could correspond to a physical quantity (temperature, density, concentration, velocity, etc.) which varies over time in a laboratory experiment. Instead of an explicit formula, we use a measurement device to capture sample values of f at predetermined points in time.

- Or, perhaps we do have a formula for $f(x)$, but this formula is not trivially easy to evaluate. Consider for example:

  $$f(x) = \sin(x) \quad \text{or} \quad f(x) = \ln(x) \quad \text{or} \quad f(x) = \int_0^x e^{-t^2} dt$$

  Perhaps evaluating $f(x)$ with such a formula is a very expensive operation and we want to consider a less expensive way to obtain a "crude approximation". In fact, in years when computers were not as ubiquitous as today, trigonometric tables were very popular. For example:

Angle	$\sin(\theta)$	$\cos(\theta)$
...	...	...
44°	0.695	0.719
45°	0.707	0.707
46°	0.719	0.695
47°	0.731	0.682
...	...	...

> If we were asked to approximate the value of $\sin(44.6°)$, it would be natural to consider deriving an estimate from these tabulated values, rather than attempting to write an analytic expression for this quantity.

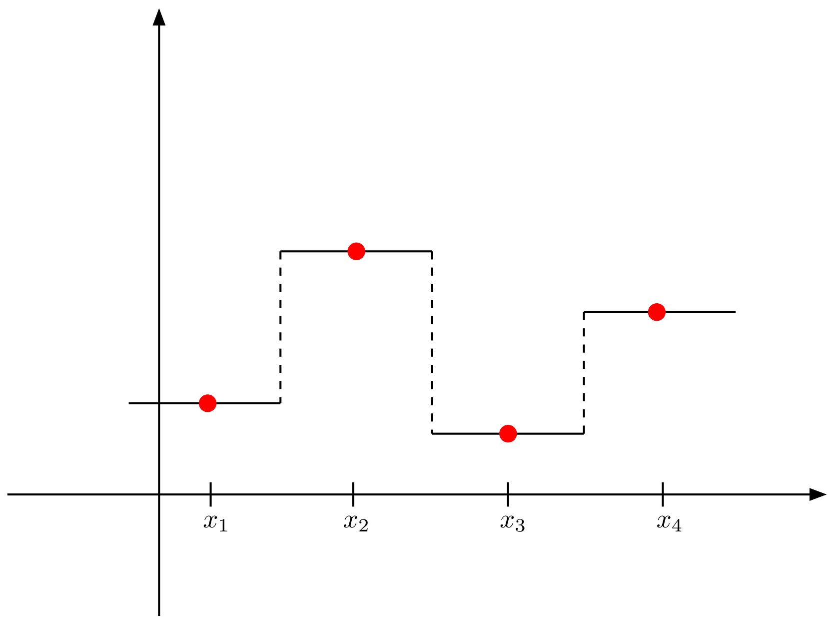
Interpolation methods attempt to answer questions about the value of $f(x)$ at points *other* than the ones it was sampled at. An obvious question would be to ask what is an estimate for $f(x^*)$ for a value x^* different than any sample we have collected; similar questions can be asked about the derivatives $f'(x^*), f''(x^*), \ldots$ at such locations.

The question of how to reconstruct a smooth function $f(x)$ that agrees with a number of collected sample values is not a straightforward one, especially since there is more than one way to accomplish this task. First, let us introduce some notation: let us write $x_1, x_2, \ldots, x_N$ for the x-locations where f is being sampled and denote the known value of $f(x)$ at $x = x_1$ as $y_1 = f(x_1)$, at $x = x_2$ as $y_2 = f(x_2)$, etc. Graphically, we seek to reconstruct a function $f(x), x \in [a, b]$ such that the plot of f passes through the following points:

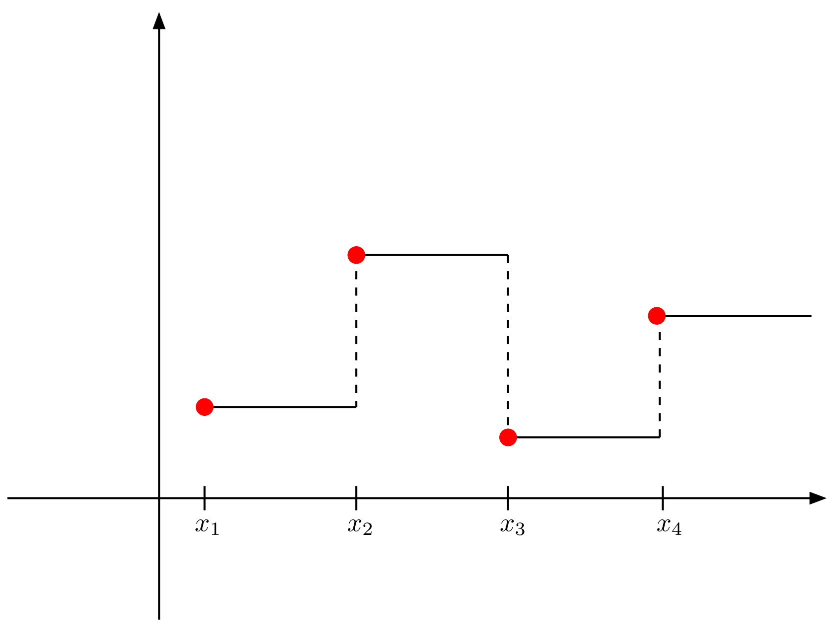
$$(x_1, y_1)\ ,\ (x_2, y_2)\ ,\ \ldots\ ,\ (x_N, y_N)$$

Here are some possible ways to do that:

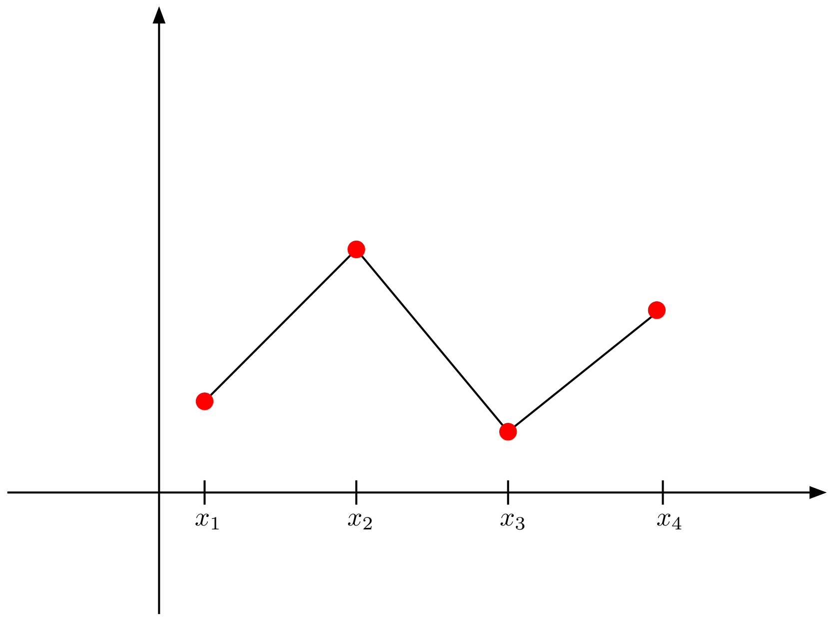
- For every x, pick the x_i closest to it, and set $f(x) = y_i$



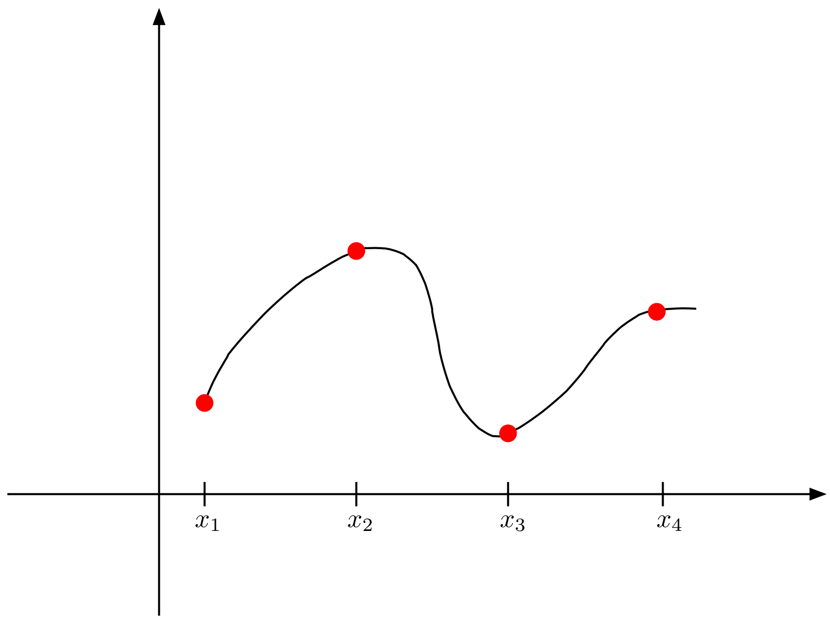
- Or, simply pick the value to the "left":



- Try connecting every two horizontally neighboring points with a straight line



- Or, try to find a smoother curve that connects them all ...



It is not trivial to argue that any particular one of these alternatives is "better", without having some knowledge of the nature of $f(x)$, or the purpose this reconstructed function will be used for. For example:

- It may appear that the discontinuous approximation generated by the "pick the closest sample" method is awkward and not as well-behaved. However, the real function $f(x)$ being sampled could have been just as discontinuous to begin with, for example, if $f(t)$ denoted the transaction amount for the customer of a bank being served at time $= t$.

- Sometimes, we may know that the real $f(x)$ is supposed to have some degree of smoothness. For example, if $f(t)$ is the position of a moving vehicle in a highway, we would expect both $f(t)$ and $f'(t)$ (velocity), possibly even $f''(t)$ (acceleration) to be continuous functions of time. In this case, if we seek to estimate $f'(t)$ at a given time, we may prefer the piecewise-linear reconstruction. If $f''(t)$ is needed, the even smoother method might be preferable.