Directly Visible Pairs and Illumination by Reflections in Orthogonal Polygons

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Abstract

We consider *direct* visibility in simple orthogonal polygons and derive tight lower and upper bounds on the number of strictly internal and external visibility edges. We also show a lower bound of $\lceil \frac{n}{2} \rceil - 1$ on the number of *diffuse* reflections required for completely illuminating an orthogonal polygon from an arbitrary point inside it. Further, we derive lower bounds on the combinatorial complexity of the *visibility polygon* of a point source S after $k \geq 1$ specular reflections within special classes of polygons.

1 Introduction

Let P be a simple polygon with n vertices. The *internal (external) visibility graph* [2] of P is a graph with vertex set equal to the vertex set of P, in which two vertices are adjacent if the line segment connecting them does not intersect the exterior (interior) of P. A visibility edge is called *strictly internal* (*strictly external*) [2] if it is not an edge of P and lies completely inside (outside) P. Line segments connecting non-consecutive vertices of the polygon that intersect polygon edges are called *mixed visibility edges* [2]. The edge bc (resp. ab) in Figure 1(i) is a strictly external (resp. internal) visibility edge, and cd is a mixed visibility edge. We focus on a special class of polygons, namely *orthogonal polygons*, in which the internal angle at each vertex is either 90 or 270 degrees.

We consider *direct* visibility and derive a lower bound of $(2n - 6)$ on the sum S of the number of strictly internal and strictly external visibility edges. We also derive an upper bound on S by counting the number of mixed visibility edges. We show these bounds to be tight by constructing two families of orthogonal polygons which achieve these bounds.

Next, we consider visibility with *reflections*. We prove that $\lceil \frac{n}{2} \rceil - 1$ *diffuse* reflections are sometimes necessary for completely illuminating a simple polygon from an arbitrary point inside it by considering a spiral orthogonal polygon. We also derive several lower bounds on the combinatorial complexity of the *visibility polygon* $V_P(S)$ of a point source S after $k \geq 1$ specular reflections within special classes of polygons P. For simple orthogonal polygons, we show that $V_P(S)$ can have $\Omega(n^2)$ *holes* with one reflection. We also show that $V_P(S)$ is simply-connected after at most two reflections in spiral orthogonal polygons, and that $V_P(S)$ can have $\Omega(n)$ holes after $\Theta(n)$ reflections in general spiral polygons.

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2 Counting visibility edges

Let $int(P)$ (resp. $ext(P)$, $mix(P)$) denote the number of strictly internal (resp. strictly external, mixed) visibility edges of a polygon P. Determining the sum of the number of strictly internal (i) and strictly external (e) visibility edges, $i + e$, of a simple polygon was posed as an open problem in [1]. This problem was settled by Urrutia in [2]. He also suggested a family of polygons (as shown in Figure 1(ii)) that achieve the bound in Theorem 1.

Theorem 1 (Urrutia [2]) *For any simple polygon P with n vertices, the number of strictly internal and strictly external visibility edges is at least $\lceil \frac{3n-1}{2} \rceil - 4$.*

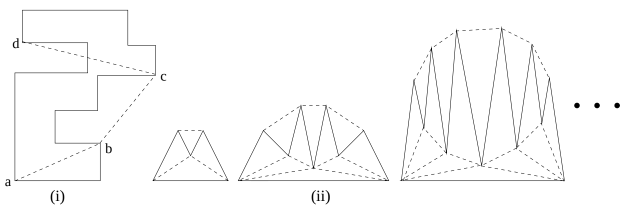

Figure 1: (i) Visibility edges. (ii) A family of simple polygons which achieve the lower bound for $(i + e)$.

2.1 Lower bound on the number of visibility edges

A partitioning of P into convex quadrilaterals is called a *convex quadrilateralization* of P. Only vertices of the polygon P may serve as vertices of the quadrilaterals. Kahn, Klawe and Kleitman [3] proved that every orthogonal polygon P (with or without holes) is convexly quadrilateralizable. Any convex quadrilateralization of P has $\frac{n-2}{2}$ convex quadrilaterals [5]. Adding a diagonal to each quadrilateral gives us a triangulation of P which has $(n - 3)$ edges. If we now flip the diagonal of each quadrilateral, we again get a triangulation of P with $\frac{n-2}{2}$ distinct edges. Using this fact to count the number of strictly internal visibility edges, we have the following lemma:

Lemma 2 *Any n-sided simple orthogonal polygon P has at least $\frac{3n-8}{2}$ strictly internal visibility edges.*

A vertex v of P is *internal* if it is inside the convex hull of P. We use Lemma 3 proved by Urrutia in [2] to derive a lower bound of $(2n-6)$ on the number of strictly internal and strictly external visibility edges.

Lemma 3 (Urrutia [2]) *If P is a simple polygon with k internal vertices, then there are at least k strictly external visibility edges, i.e., $ext(P) \geq k$.*

Theorem 4 *There are at least $(2n-6)$ strictly internal and strictly external visibility edges in any simple orthogonal polygon P with n vertices.*

Proof. First note that all reflex vertices of P are internal vertices. O' Rourke showed that there are $\frac{n-4}{2}$ reflex vertices in any orthogonal polygon with n vertices [5]. So from Lemma 3, there are at least $\frac{n-4}{2}$ strictly external visibility edges in P. Using this fact and Lemma 2, we get $int(P) + ext(P) \geq 2n-6$. □

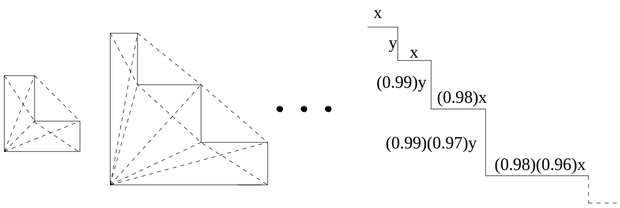

Figure 2: (a) Staircase polygons which achieve the lower bound of $(2n-6)$. (b) Construction scheme for the staircase (x and y are any two positive integers).

We show this bound to be tight by constructing a family of staircase polygons for which $i + e = 2n-6$ (see Figure 2(a)). A staircase polygon is an isothetic polygon bounded by two monotonically rising (falling) staircases. The staircase is constructed as shown in Figure 2(b).

2.2 Upper bound on the number of visibility edges

In simple polygons, it is easy to see that $int(P) + ext(P) \leq ({}^nC_2 - n)$. This bound is achieved by a convex polygon where the number of mixed visibility edges is zero. However, $mix(P)$ is never zero for orthogonal polygons. We call a horizontal edge of P a *top (bottom)* edge if $int(P)$ is below (above) it. Similarly, we define *left (right)* vertical edges of P. We derive the following upper bound and also show it to be tight by constructing a family of staircase polygons. See Figure 4 for the construction.

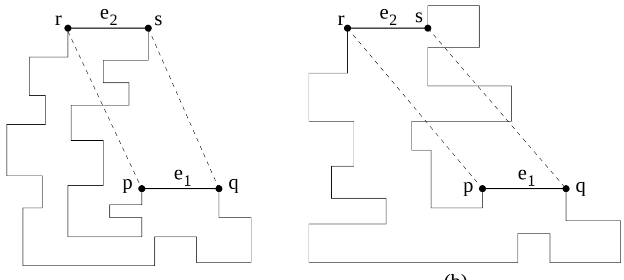

Figure 3: A mixed visibility edge is present between any two top edges (symmetric cases are also possible).

Theorem 5 *There are at most $\frac{n(n-3)}{2} - \sum_{i=1}^{4} \frac{n_i(n_i-1)}{2}$ strictly internal and strictly external visibility edges in any simple orthogonal polygon P with n vertices, where n_1 (resp. n_2, n_3, n_4), is the number of top (resp. bottom, left, right) edges in P.*

Proof. Let e_1 and e_2 be any two top edges in P with e_1 lying below e_2. Let p and q (resp. r and s) be the end-points of e_1 (resp. e_2). See Figure 3. Consider the quadrilateral Q formed by joining p to r and q to s, where traversal of the boundary $bd(P)$ of P in an anticlockwise fashion starting from p gives the sequence of the vertices visited as $p \to s \to r \to q$. Since rs is a top edge, there exists a point x inside Q infinitesimally below rs lying inside P. Similarly, there exists a point y inside Q infinitesimally above pq lying outside P. We conclude that $bd(P)$ intersects the interior of Q. So at least one of the four edges of Q must be a mixed visibility edge. So there is at least one mixed visibility edge corresponding to every pair of top edges. Symmetrically, this claim also holds for every pair of left (resp. right, bottom) edges also. From the above observation we conclude that, $mix(P) \geq \sum_{i=1}^{4} \frac{n_i(n_i-1)}{2}$. Using this observation and the fact that $int(P) + ext(P) + mix(P) = \frac{n(n-1)}{2} - n$, we get $int(P) + ext(P) \leq \frac{n(n-3)}{2} - \sum_{i=1}^{4} \frac{n_i(n_i-1)}{2}$. □

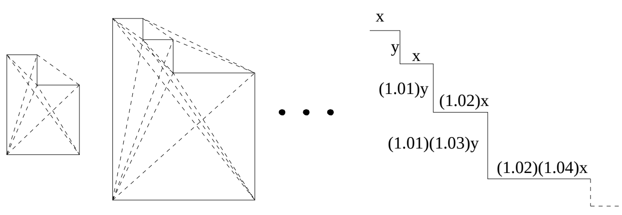

Figure 4: (a) Staircase polygons which achieve the upper bound. (b) Construction scheme for the staircase (x and y are any two positive integers).

3 Visibility with reflections

The problem of visibility when *reflections* from the interior of edges are allowed was first considered in [6, 7, 8]. Two kinds of reflections were defined, *specular*, in which Newton's laws of reflection are obeyed, and *diffuse*, in which light is reflected back in *all* possible directions over a spread of 180 degrees. We denote the *visibility polygon* [4] of a point source S in a simple polygon P by $V_P(S)$.

3.1 Illumination with diffuse reflections

An interesting problem in visibility is to bound the number of diffuse reflections required for completely illuminating a given polygon. The portion m of an edge e that becomes visible at the kth reflection is called a *mirror* at the kth stage. The union of mirrors at the kth stage of reflection, lying on an edge e of P, form connected components called *reflecting segments*. Prasad et al. [8] proved the following lemma:

Lemma 6 ([8]) *Let the edge e of P have a reflecting segment at the lth stage of reflection. Then, the entire edge e is a reflecting segment at the $(l+2)$nd stage.*

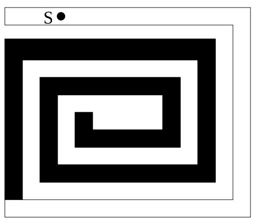

Figure 5: Illumination after two diffuse reflections in spiral polygons. (Illuminated regions are shaded white, non-illuminated regions are shaded black.)

Consider an n-sided spiral orthogonal polygon Q. We claim that $\lceil \frac{n}{2} \rceil - 1$ reflections are necessary for completely illuminating Q from any arbitrary point inside it. Using Lemma 6 one can show that starting from the *arm* containing the source S light successively floods adjoining arms, thereby propagating inside the polygon. So each *arm* of Q requires one diffuse reflection to become illuminated. Hence, Q becomes completely illuminated after $\lceil \frac{n}{2} \rceil - 1$ diffuse reflections. It is well-known that the entire polygon will be illuminated after n diffuse reflections [9]. However, we believe that $\lceil \frac{n}{2} \rceil - 1$ diffuse reflections are also **sufficient** for complete illumination. We have not been able to prove it as yet.

One approach for proving this result would be to study diffuse reflections in the light of the *cooperative guards* problem (see [10]). However, the guards chosen should be *edge guards* instead of *vertex guards* since the latter can see "around the corner", which is not permitted in reflections.

3.2 Combinatorial complexity of visibility polygons with specular reflections

Let P be any n-sided simple polygon. A *blind spot* is a connected component of $P \backslash V_P(S)$. *Holes* are blind spots which do not intersect the boundary of P. Aronov et al. [7] proved that the visibility polygon $V_P(S)$ of a point source S has complexity $\mathcal{O}(n^2)$ with one specular reflection and that this bound is tight. We show that this bound can also be achieved in case of orthogonal polygons, even though the topology of such polygons does not permit the initial angle of incidence of a ray to change significantly. See Figure 6 for an example with $\Omega(n^2)$ holes.

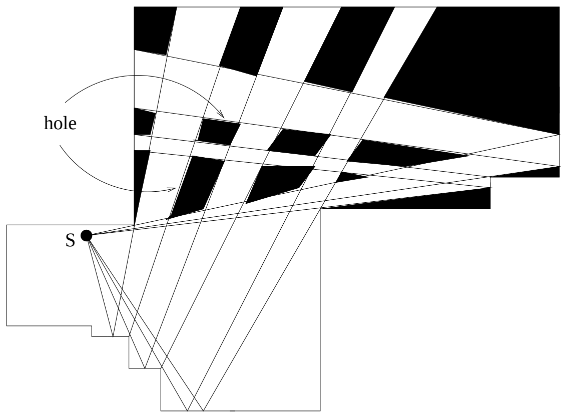

Figure 6: $\Omega(n^2)$ holes with one specular reflection.

Now consider spiral orthogonal polygons. For at most two reflections, we show that $V_P(S)$ is simply-connected in such polygons. For one specular reflection, a *vpath limit* is defined as any 2-link path in P from S to $bd(P)$ that obeys the reflection property but passes through a vertex of P. Blind spots are created by the intersection of vpath limits. Each link of a vpath is given a direction which is same as that of the ray defining it. Suppose two vpath links cross at x. Removal of these two links produces four or more *quadrants*, which are classified as left, right, bottom, or top. If the blind spot lies in the left (right, bottom, top) quadrant with respect to x, locally near x, then x is called a *left* (*right*, *bottom*, *top*) vertex. See [7] for details on the above definitions. The following result was proved in [7]:

Lemma 7 ([7]) *A hole is a closed convex polygon with each vertex on two vpath limits; it contains exactly one top and one bottom vertex.*

Lemma 7 can be generalized to multiple specular reflections. Using this result, one can show that at least three sources (real or virtual), which throw light in a shadow, are required for creating a hole at the kth stage of reflection. See Figure 7. There exist only two sources after both one and two specular reflections.

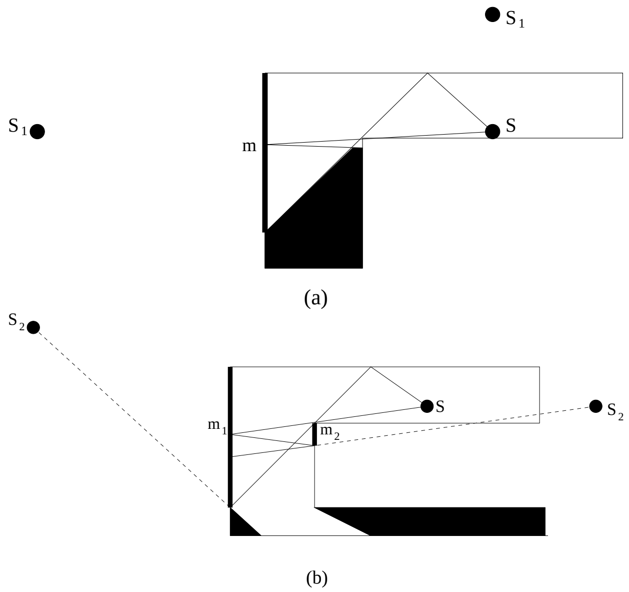

Figure 7: $V_P(S)$ is simply connected after one and two specular reflections.

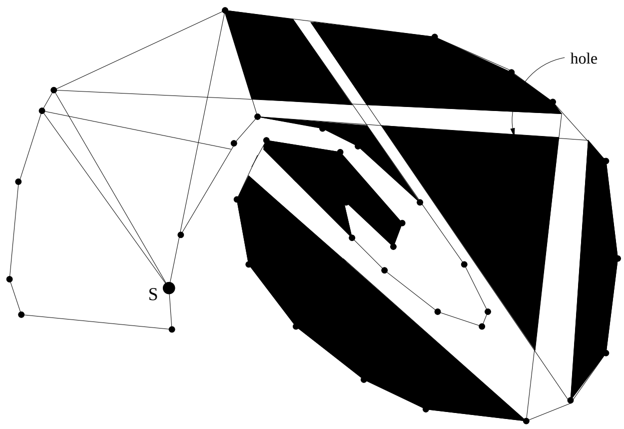

Figure 8: A hole after three specular reflections.

So $V_P(S)$ remains simply-connected. However, after $k \geq 3$ specular reflections, we observed that the analysis became extremely complicated. We are unaware of the combinatorial complexity of $V_P(S)$ after $k \geq 3$ specular reflections. In case of general spiral polygons however, we observed that $V_P(S)$ is multiply-connected after three specular reflections (see Figure 8). Note that the construction shown in Figure 8 isolates a single beam of light in each *curl* of the spiral polygon. The part of the beam reflected back in the previous curl can be made infinitely thin, so that it has no subsequent effect on the connectedness of $V_P(S)$. We can extend this construction by *curling* the polygon $\Theta(n)$ times to prove the following result:

Theorem 8 *The visibility polygon $V_P(S)$ of a point source S can have $\Omega(n)$ holes in an n-sided spiral polygon P after $\Theta(n)$ specular reflections.*

4 Conclusion

We derived tight lower and upper bounds on the complexity of the internal and external visibility graphs of orthogonal polygons. We considered the problem of illuminating a polygon by multiple diffuse reflections and also studied the complexity of the visibility polygon $V_P(S)$ of a point source S after multiple specular reflections within special classes of polygons. Apart from the several open problems discussed in the paper, we are also studying the complexity of $V_P(S)$ after multiple diffuse reflections. We believe that $V_P(S)$ is always simply-connected in orthogonal polygons. However, we still lack a proof. We also believe that it might be possible to solve the following conjecture stated in [8] for orthogonal polygons, and we are currently working towards proving this result.

Conjecture 1 ([8]) *The total complexity of $V_P(S)$ after k diffuse reflections is $\Theta(n^2)$.*

Acknowledgements

The first author would like to thank Monique Teillaud and Olivier Devillers of INRIA, Sophia Antipolis, France for helpful discussions on the topic.

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