Graph Isomorphism for Colored Graphs with Color Multiplicity Bounded by 3

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The colored graph isomorphism problem is a restricted version of the general graph isomorphism (GI) problem that involves deciding the existence of a color preserving isomorphism between a pair of colored graphs. In this report, we study this problem for graphs whose color multiplicity is bounded by 3 (3-GI). We begin by formally defining the colored graph isomorphism problem and setup our notation. Next, we study the general graph isomorphism problem and specialize its results for colored graphs. Then, we use special properties of colored graphs with multiplicity bounded by 3 to prove that 3-GI is in the deterministic-logarithmic-space complexity class L. Finally, we use this result to show that the problem of deciding the existence of a non-trivial color preserving automorphism on a graph with color multiplicity bounded by 3 (3-GA) is in L as well. In previous work [1], a proof of 3-GI in symmetric logspace SL (which we now know to be the same as L) has been given but the proof is incorrect and section 4 presents a counter example.

Definition 1. A *colored* graph G is a three-tuple (V, E, P) where (V, E) specifies an undirected graph and $P = \{V_i\}_{i=1}^k$ is a partition of the vertices into color sets ($V_i \cap V_j = \Phi, i \neq j$ and $(\cup_{i=1}^k V_i) = V$). For convenience, define $color(v) = i$ if $v \in V_i$.

Definition 2 (Colored GI). The colored graph isomorphism problem is to decide the existence of a color preserving isomorphism between a pair of colored graphs $G = (V, E, P)$ and $G' = (V', E', P')$, i.e., a mapping $\phi : V \rightarrow V'$ satisfying the following conditions.

1. ϕ is a bijection.
2. $(v_1, v_2) \in E \Leftrightarrow (\phi(v_1), \phi(v_2)) \in E'$ for all $v_1, v_2 \in V'$.
3. $color(v) = color(\phi(v))$ for all $v \in V$.

Also, let $\mathrm{Iso}(G, G')$ denote the set of isomorphisms between graphs G and G' and $\mathrm{Iso}_c(G, G')$ denote the set of color preserving isomorphisms between colored graphs G and G'.

1 Decomposing isomorphisms

For a graph $G = (V, E)$ and a set $V' \subseteq V$, let $G_{V'}$ denote the graph induced by vertices V' and let $E_{V'}$ denote the edges of $G_{V'}$. Also, for a subset of the edges $E' \subseteq E$, let $G - E'$ be the graph G without the edges in E'. Next, if $\phi_1 : V_1 \rightarrow V_1'$ and $\phi_2 : V_2 \rightarrow V_2'$ are bijections on disjoint domains ($V_1 \cap V_2 = \Phi$) and disjoint co-domains ($V_1' \cap V_2' = \Phi$) then let $\phi_1 \times \phi_2 : (V_1 \cup V_2) \rightarrow (V_1' \cup V_2')$ be the bijection defined as follows.

$$\phi_1 \times \phi_2(v) = \begin{cases} \phi_1(v) \text{ if } v \in V_1 \\ \phi_2(v) \text{ if } v \in V_2 \end{cases}$$

The following lemma shows how we can use isomorphisms on smaller graphs to find isomorphisms on larger graphs. The idea is that the edge-preserving condition of a graph isomorphism can be satisfied locally by isomorphisms on smaller edge-disjoint subgraphs.

Lemma 1. Given a pair of graphs $G = (V, E)$ and $G' = (V', E')$ and partitions $P = \{V_i\}_{i=1}^k$ of V and $P' = \{V_i'\}_{i=1}^k$ of V', if there exist bijections $\phi_i : V_i \rightarrow V_i'$ for $1 \leq i \leq k$ such that $\phi_i \times \phi_j \in \mathrm{Iso}(G_{V_i \cup V_j}, G'_{V_i' \cup V_j'})$ for all $1 \leq i < j \leq k$ then $\phi_1 \times \phi_2 \times \cdots \times \phi_k \in \mathrm{Iso}(G, G')$.

Proof. Let $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_k$. Then

$$\begin{array}{lll} & (v_1, v_2) \in E & \\ \Leftrightarrow & (v_1, v_2) \in E_{V_a \cup V_b} & \text{where } v_1 \in V_a,\ v_2 \in V_b \\ \Leftrightarrow & (\phi_a \times \phi_b(v_1), \phi_a \times \phi_b(v_2)) \in E'_{V'_a \cup V'_b} & (\text{as } \phi_a \times \phi_b \in \text{Iso}(G_{V_a \cup V_b}, G'_{V'_a \cup V'_b})) \\ \Leftrightarrow & (\phi_a(v_1), \phi_b(v_2)) \in E'_{V'_a \cup V'_b} & \\ \Leftrightarrow & (\phi(v_1), \phi(v_2)) \in E'_{V'_a \cup V'_b} & \\ \Leftrightarrow & (\phi(v_1), \phi(v_2)) \in E' & \diamondsuit \end{array}$$

The conditions required in Lemma 1 can be further simplified using the following Lemma. The idea is to again find smaller edge-disjoint subgraphs.

Lemma 2. Given a pair of graphs $G = (V, E)$ and $G' = (V', E')$; disjoint subsets $V_1, V_2 \subseteq V$ and $V'_1, V'_2 \subseteq V'$ and bijections $\phi_1 : V_1 \to V'_1$ and $\phi_2 : V_2 \to V'_2$, then $\phi_1 \times \phi_2 \in \text{Iso}(G_{V_1 \cup V_2}, G'_{V'_1 \cup V'_2})$ if and only if $\phi_1 \in \text{Iso}(G_{V_1}, G'_{V'_1})$, $\phi_2 \in \text{Iso}(G_{V_2}, G'_{V'_2})$ and $\phi_1 \times \phi_2 \in \text{Iso}(G_{V_1 \cup V_2} - (E_{V_1} \cup E_{V_2}), G'_{V'_1 \cup V'_2} - (E'_{V'_1} \cup E'_{V'_2}))$.
Proof. The proof is similar to the one for Lemma 1. $\diamondsuit$

Using the above two Lemmas, we get the following important property about color preserving isomorphisms.

Theorem 1. Given a pair of colored graphs $G = (V, E, \{V_i\}_{i=1}^k)$ and $G' = (V', E', \{V'_i\}_{i=1}^k)$, $\phi \in \text{Iso}_c(G, G')$ if and only if ϕ can be written as $\phi_1 \times \phi_2 \times \cdots \times \phi_k$ where

1. $\phi_i \in \text{Iso}(G_{V_i}, G'_{V'_i}),\ 1 \le i \le k$ and
2. $\phi_i \times \phi_j \in \text{Iso}(G_{V_i \cup V_j} - (E_{V_i} \cup E_{V_j}), G'_{V'_i \cup V'_j} - (E'_{V'_i} \cup E'_{V'_j})),\ 1 \le i < j \le k$.

Proof. Let $\phi \in \text{Iso}_c(G, G')$. Then we must have $color(v) = color(\phi(v))$ for all $v \in V$. This implies the following.

$$\begin{array}{ll} & v \in V_i \\ \Leftrightarrow & color(\phi(v)) = i \\ \Leftrightarrow & \phi(v) \in V'_i \end{array}$$

Now, define $\phi_i : V_i \to V'_i$ as simply ϕ restricted to the domain V_i. Then clearly, $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_n$. Also, since ϕ is an isomorphism, we have $\phi_i \times \phi_j \in \text{Iso}(G_{V_i \cup V_j}, G'_{V'_i \cup V'_j}),\ 1 \le i < j \le k$. Using Lemma 2 on this, we get the forward direction of this theorem. For the reverse direction, use Lemma 2 to get that $\phi_i \times \phi_j \in \text{Iso}(G_{V_i \cup V_j}, G'_{V'_i \cup V'_j}),\ 1 \le i < j \le k$. Then use Lemma 1 to get that $\phi \in \text{Iso}(G, G')$. Then ϕ can be shown to preserve colors as each ϕ_i is color preserving by condition 1. $\diamondsuit$
Corollary 1. A pair of colored graphs $G = (V, E, \{V_i\}_{i=1}^k)$ and $G' = (V', E', \{V'_i\}_{i=1}^k)$ are isomorphic if and only if there are bijections $\phi_i : V_i \to V'_i$ satisfying conditions 1 and 2 mentioned in Theorem 1.

Corollary 2. A colored graph $G = (V, E, \{V_i\}_{i=1}^k)$ has a non-trivial (color preserving) automorphism if and only if there are bijections $\phi_i : V_i \to V_i$ satisfying conditions 1 and 2 mentioned in Theorem 1 (with $G' = G$) along with an additional constraint that at least one of the bijection ϕ_i is not the identity mapping.

2 3-GI is in L

In this section, we show that 3-GI is in the complexity class L by reducing problem instances of 3-GI in logspace to undirected graph reachability queries. So, given a pair of colored undirected graphs $G = (V, E, \{V_i\}_{i=1}^n)$ and $G' = (V', E', \{V'_i\}_{i=1}^n)$ with color multiplicity bounded by 3 (called 3-colored graphs: $|V_i| \le 3$ and $|V'_i| \le 3$ for all $1 \le i \le n$), we will construct in logspace a single undirected graph $f(G, G')$ and pose the isomorphism question as a reachability question on this graph. Before we describe the construction of $f(G, G')$, let us examine theorem 1 in greater detail for 3-colored graphs.
Condition (1) of theorem 1 considers isomorphisms between the graphs G_{V_i} and $G'_{V'_i}$ both of which have bounded size (≤ 3 vertices) for 3-colored graphs G and G'. So, we can enumerate $\text{Iso}(G_{V_i}, G'_{V'_i})$ in

constant time by considering all the bijections between V_i and V_i'. If we define $B(V_i, V_i')$ to be all the bijections between V_i and V_i', condition (1) can be posed as multiple constraints of the form

$$(\alpha) \quad \phi_i \neq \pi$$

where $\pi \in B(V_i, V_i') - \mathrm{Iso}(G_{V_i}, G'_{V_i'})$. Note that there can be at most 6 such constraints per color i. The reason why we have chosen to pose complementary constraints will become clear later when we construct the graph $f(G, G')$. Next, condition (2) could also be broken down into constraints of the following form through exhaustive case analysis.

$$\begin{array}{ll} (\alpha) & \phi_i \neq \pi \\ (\alpha) & \phi_j \neq \pi' \\ (\beta) & \phi_i = \pi_i \Leftrightarrow \phi_j = \pi_j \end{array}$$

where $\pi, \pi_i \in B(V_i, V_i')$ and $\pi', \pi_j \in B(V_j, V_j')$. To further simplify this discussion, let $(\alpha_{i,\pi})$ denote the constraint $\phi_i \neq \pi$ and $(\beta_{i,j,\pi,\pi'})$ denote the constraint $\phi_i = \pi \Leftrightarrow \phi_j = \pi'$. These constraints are over variables ϕ_i which together form an isomorphism between given graphs G and G'. πs denote fixed bijections. Using theorem 1 we can say that any color preserving isomorphism has to be composed of bijections ϕ_i that satisfy all the generated constraints and vice versa. An exhaustive case analysis is provided in the appendix to find out all constraints but we present one case here for the continuity of this section. Figure 2 shows the constraints generated for graphs shown in figure 1.

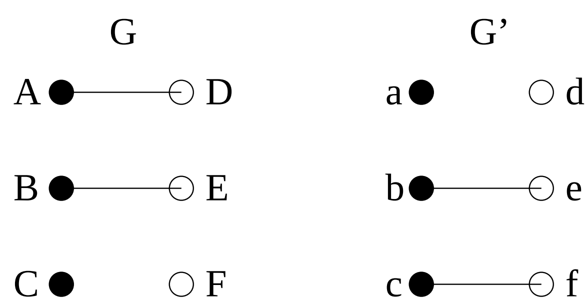


Figure 1: Colored graphs G and G' with two colors. Vertices $\{A, B, C, a, b, c\}$ and $\{D, E, F, d, e, f\}$ have the same color

$$\begin{array}{ll} (\alpha) & \phi_1 \notin \{abc, acb, bac, cab\} \\ (\alpha) & \phi_2 \notin \{def, dfe, edf, fde\} \\ (\beta) & \phi_1 = bca \Leftrightarrow \phi_2 = efd \\ (\beta) & \phi_1 = cba \Leftrightarrow \phi_2 = fed \end{array}$$

Figure 2: Constraints generated for the colored graph pair shown in figure 1. For ϕ_1, abc is a shorthand for $\{A \mapsto a, B \mapsto b, C \mapsto c\}$. Similarly for ϕ_2.

Another benefit obtained from 3-colored graphs is the restriction on the constraints that can be produced. In particular, we have the following property on the type of constraints we can have.

Lemma 3. For 3-colored graphs G and G' (as defined in Theorem 1), if there is a constraint $(\beta_{i,j,\pi,\pi'})$ then,

1. For all $\psi_i \in B(G_{V_i}, G'_{V_i'})$, either (α_{i,ψ_i}) is a constraint or $(\beta_{i,j,\psi_i,\pi''})$ is a constraint for some $\pi'' \in B(G_{V_j}, G'_{V_j'})$.
2. For all $\psi_j \in B(G_{V_j}, G'_{V_j'})$, either (α_{j,ψ_j}) is a constraint or $(\beta_{i,j,\pi'',\psi_j})$ is a constraint for some $\pi'' \in B(G_{V_i}, G'_{V_i'})$.

Proof. Clear from the case analysis provided in the appendix. $\diamondsuit$

This Lemma basically says that whenever a (β) constraint is produced between two colors for some pair of bijections, there would be (β) constraints between the colors for all bijections that do not have an (α) constraint. This property will clearly show up when we construct the graph $f(G, G')$. Another important inference we can make out from the generated constraints is the following.

Lemma 4. For 3-colored graphs G and G' (as defined in Theorem 1), $\pi_i \in \text{Iso}(G_{V_i}, G'_{V'_i})$ if there are no (α_{i,π_i}) constraints and $\pi_i \times \pi_j \in \text{Iso}(G_{V_i \cup V_j}, G'_{V'_i \cup V'_j})$ if

1. There are no (α_{i,π_i}) and (α_{j,π_j}) constraints and
2. There is the constraint $(\beta_{i,j,\pi_i,\pi_j})$ or there are no $(\beta_{i,j,\_,\_})$ constraints.

The above Lemma shows how the (α) and (β) constraints capture the information required to determine isomorphisms for a pair of colors. Now we will use the constraints to generate the graph $f(G, G')$.

2.1 The graph $f(G, G')$

Given 3-colored graphs $G = (V, E, \{V_i\}_{i=1}^k)$ and $G' = (V', E', \{V'_i\}_{i=1}^k)$, construct the graph $f(G, G')$ as follows. The vertices of $f(G, G')$ are two-tuples of the form (i, π) where $1 \leq i \leq k$ is a color and $\pi \in B(V_i, V'_i)$ is a bijection. There is one additional vertex in $f(G, G')$ denoted by $\perp$. The edges of this graph are as follows.

1. $(\perp, (i, \pi))$ is an edge if $(\alpha_{i,\pi})$ is a constraint.
2. $((i, \pi), (j, \pi'))$ is an edge if $(\beta_{i,j,\pi,\pi'})$ is a constraint.

So, the edges of $f(G, G')$ directly encode the constraints. We can now use graph reachability as a resolution procedure for the constraints and find the required color preserving isomorphisms. Let $C_G(v)$ denote the connected component of a vertex v in the graph G. We drop the subscript G when the graph is clear from the context.

Theorem 2. 3-colored graphs G and G' are isomorphic if and only if for all colors i, there is a vertex $v = (i, \pi_i)$ in $f(G, G')$ s.t.

1. $\perp \notin C(v)$ ($\perp$ is not reachable from v).
2. $(j, \pi) \in C(v) \Rightarrow (j, \pi') \notin C(v)$ for all $\pi' \neq \pi$ (No two vertices of the same color are reachable from v).

Each of the conditions in the above Theorem can be translated into graph reachability queries on $f(G, G')$. Thus, by running reachability queries for each vertex in $f(G, G')$ (except $\perp$) and keeping track of the fact that we were able to find a vertex of each explored color that satisfied the conditions of the Theorem, we can find out if G and G' are isomorphic or not. This problem can in turn be converted to a single reachability question ($\text{L}^{\text{L}} = \text{L}$) which shows that 3-GI is in L. So, we are only left with the proof of Theorem 2 which we state after the following useful Lemmas.

Lemma 5. If $\{v_l = (i_l, \pi_l)\}_{l=1}^m$ is a path in the graph $f(G, G')$ which does not include the vertex $\perp$, then we have the implied constraint $(\beta_{i_1,i_m,\pi_1,\pi_m})$.
Proof. This follows straight from induction. The constraints $(\beta_{i_1,i_2,\pi_1,\pi_2})$ and $(\beta_{i_2,i_3,\pi_2,\pi_3})$ clearly imply the constraint $(\beta_{i_1,i_3,\pi_1,\pi_3})$. $\diamondsuit$

Lemma 6. Connected components of $f(G, G')$ that don't contain $\perp$ either have the same colors or completely different ones. Formally, if $\mathcal{C}(S) = \{i \mid \exists \pi_i,\ (i, \pi_i) \in S\}$ then for any two vertices v and v' in $f(G, G')$ that are not reachable from $\perp$, $\mathcal{C}(C(v)) \cap \mathcal{C}(C(v')) = \Phi$ or $\mathcal{C}(C(v)) = \mathcal{C}(C(v'))$.
Proof. Suppose that this is not true for two vertices v and v' that are not reachable from $\perp$. Then *wlog* $\mathcal{C}(C(v)) \cap \mathcal{C}(C(v')) \neq \Phi$ and $\mathcal{C}(C(v)) - \mathcal{C}(C(v')) \neq \Phi$. Now choose an edge in $C(v)$ whose one vertex has color in $\mathcal{C}(C(v)) - \mathcal{C}(C(v'))$ and the other vertex has color in $\mathcal{C}(C(v)) \cap \mathcal{C}(C(v'))$. Such

an edge would always exist because of the connectivity of $C(v)$. Let this edge be $((j,\pi_1),(i,\pi_2))$ with $i \in \mathcal{C}(C(v)) \cap \mathcal{C}(C(v'))$. Then there is a vertex $(i,\pi_3) \in C(v')$. Now apply Lemma 3. Since there is a constraint $(\beta_{i,j,\pi_2,\pi_1})$, there should also be a constraint $(\beta_{i,j,\pi_3,\pi_4})$ for some π_4 as there is no (α_{i,π_3}) constraint ($C(v')$ does not have $\perp$). So, there should be a j colored vertex in $C(v')$ which is not true. This contradiction proves this Lemma. $\diamondsuit$

We are now setup to prove Theorem 2.

Proof of Theorem 2. First, suppose that G and G' are isomorphic and $\pi \in \mathrm{Iso}_c(G,G')$. Then by Theorem 1 we have $\pi = \pi_1 \times \pi_2 \times \cdots \times \pi_k$, $\pi_i \in \mathrm{Iso}(G_{V_i}, G'_{V'_i})$ and $\pi_i \times \pi_j \in \mathrm{Iso}(G_{V_i \cup V_j} - (E_{V_i} \cup E_{V_j}), G'_{V'_i \cup V'_j} - (E'_{V'_i} \cup E'_{V'_j}))$. Now, consider $C((i,\pi_i))$. If $\perp \in C((i,\pi_i))$ then there is a path from (i,π_i) to some vertex (j,π') which is connected to $\perp$. This means that there are constraints $(\beta_{i,j,\pi_i,\pi'})$ (using Lemma 5) and $(\alpha_{j,\pi'})$. These constraints imply $\phi_i \neq \pi_i$ for any isomorphism that agrees with π_i on color i which is a contradiction. Hence, $\perp \notin C((i,\pi_i))$. Now, if $(j,\pi'),(j,\pi'') \in C((i,\pi_i))$ for $\pi' \neq \pi''$, then there are paths $(i,\pi_i) \rightsquigarrow (j,\pi')$ and $(i,\pi_i) \rightsquigarrow (j,\pi'')$. Using Lemma 5 again, we have the constraints $(\beta_{i,j,\pi_i,\pi'})$ and $(\beta_{i,j,\pi_i,\pi''})$. These imply that any isomorphism that agrees with π_i on color i must agree with both π' and π'' on color j which is not possible. However, an isomorphism does exist that agrees with π_i on color i. Hence we have a contradiction and the vertex (i,π_i) satisfies the required properties for Theorem 2 for all colors i.
For the reverse direction, we have at least one vertex v_i for each color i whose connected component $C(v_i)$ satisfies the mentioned properties. Now, use algorithm 1 to choose some of these vertices (at most one per color) such that their connected components are color disjoint but span all the colors.

Algorithm 1: Choosing vertices
$CC \leftarrow \Phi$
$CV \leftarrow \Phi$
while $(CC \neq \{1,\cdots,k\})$
  Choose any vertex v_j s.t. $j \notin CC$ and $C(v_j)$ satisfies properties of Theorem 2.
  $CV \leftarrow CV \cup C(v_j)$
  $CC \leftarrow CC \cup \mathcal{C}(C(v_j))$
return CV

Lemma 6 ensures that the connected components added to CV are disjoint with respect to each other. Also, since each of $C(v_i)$ contains at most one vertex of each color and doesn't contain $\perp$, CV contains exactly one vertex of each color and does not contain $\perp$. So, let $CV = \{(i,\pi_i) \mid 1 \leq i \leq k\}$. Now, we will show that $\pi = \pi_1 \times \pi_2 \times \cdots \times \pi_k$ is a color preserving isomorphism. For this, we only need to show that $\pi_i \times \pi_j \in \mathrm{Iso}(G_{V_i \cup V_j}, G'_{V'_i \cup V'_j})$, $1 \leq i < j \leq k$. Lemma 2 and Theorem 1 complete the proof after that. Now, if (i,π_i) and (j,π_j) lie in different connected components then there is no edge in $f(G,G')$ between vertices of colors i and j as an edge would imply a (β) constraint between the colors which in turn implies that there is a constraint $(\beta_{i,j,\pi_i,\pi'})$ for some π' (because of Lemma 3). Since the connected components were color disjoint, a vertex of color j cannot exist in the connected component of (i,π_i). Hence there is no edge between the colors. The absence of such an edge along with the absence of edges to $\perp$ from (i,π_i) and (j,π_j) implies that $\pi_i \times \pi_j \in \mathrm{Iso}(G_{V_i \cup V_j}, G'_{V'_i \cup V'_j})$ (using Lemma 4). Now suppose that (i,π_i) and (j,π_j) lie in the same connected component. Since each connected component had at most one vertex per color, we have that either (i,π_i) is connected to (j,π_j) or there is no edge between the two colors using an argument similar to the one presented before for (i,π_i) and (j,π_j) lying in different connected components. Again, using Lemma 4, $\pi_i \times \pi_j \in \mathrm{Iso}(G_{V_i \cup V_j}, G'_{V'_i \cup V'_j})$. This completes the proof to Theorem 2.

3 3-GA is in L

We can reduce 3-GA into 3-GI queries to prove that 3-GA is in L (using the same reduction as the many-one reduction of GA into GI) but we can also directly show this by following the above technique. For a 3-colored graph $G = (V, E, \{V_i\}_{i=1}^k)$, construct the graph $f(G,G)$ and then pose the following question for determining the existence of a non-trivial automorphism: for all colors i, there should be a vertex $v = (i, \pi_i)$ in $f(G, G)$ s.t.

1. $\bot \notin C(v)$.
2. $(j, \pi) \in C(v) \Rightarrow (j, \pi') \notin C(v)$ for all $\pi' \neq \pi$.

Also, at least one of these vertices v should be something other than (i, id) where id is the identity bijection.

4 Previous Work

In a previous paper by Koebler and Toran [1], a proof has been presented for 3-GI being in L but it has a flaw. The proof proceeds by building the graph $f(G, G')$ (on the same vertex set as we have used) and then converts 3-GI in to graph reachability queries on this graph. The construction of $f(G, G')$ is based on the following rules. ($G = (V, E, \{V_i\}_{i=1}^k)$ and $G' = (V', E', \{V_i'\}_{i=1}^k)$)

Rule 1. For every pair of colors i, j and every bijection $\phi \in B(V_i, V_j)$, include the edge $((i, \phi), \bot)$ in $f(G, G')$ if the edges between V_i and V_j in G and the edges between V_i' and V_j' in G' imply that no isomorphism in $\text{Iso}(G, G')$ can map nodes of V_i into V_i' like ϕ, that is, for every $\psi \in B(V_j, V_j')$, $\phi \times \psi \notin \text{Iso}(G_{V_i \cup V_j}, G'_{V_i' \cup V_j'})$.

Rule 2. For every pair of colors i, j, and $\phi \in B(V_i, V_i')$, if for a pair of nodes $i_a, i_b \in V_i = \{i_1, i_2, i_3\}$ and a pair of nodes $i'_{a'}, i'_{b'} \in V_i' = \{i'_1, i'_2, i'_3\}$ and two bijections η, π on the set $\{1, 2, 3\}$, the edges between the sets of nodes $\{i_a, i_b\}$ and $\{j_{\eta(a)}, j_{\eta(b)}\}$ are exactly the two edges $\{(i_a, j_{\eta(a)}), (i_b, j_{\eta(b)})\}$ and the edges between the sets of nodes $\{i'_{a'}, i'_{b'}\}$ and $\{j'_{\pi(a')}, j'_{\pi(b')}\}$ are exactly the two edges $\{(i'_{a'}, j'_{\eta(a')}), (i'_{b'}, j'_{\eta(b')})\}$ and $\phi \times \psi \in \text{Iso}((G_{V_i \cup V_j}, G'_{V_i' \cup V_j'})$ for $\psi = \pi\phi\eta^{-1}$ then we include in $f(G, G')$ the edge $((i, \phi), (j, \psi))$.

After constructing the graph $f(G, G')$, isomorphism of G and G' is decided by looking for a set of nodes in $f(G, G')$, one of each color, such that from this set no other node in $f(G, G')$ can be reached. This result is based on the following Lemma which is the first place where the proof fails.

Lemma 4 of [1]. For each pair of colors i, j and bijection ϕ, ψ, if there is a path from (i, ϕ) in $f(G, G')$ to (j, ψ) not having $\bot$ as an intermediate node then every isomorphism in $\text{Iso}(G, G')$ that maps the nodes of color i like ϕ, is forced to map the node of color j like ψ.

The proof for base case of this Lemma is incorrect. It assumes that if there is an edge between (i, ϕ) and (j, ψ) then $\psi = \pi\phi\eta^{-1}$ for some bijections η and π satisfying conditions of **Rule 2** of the graph construction. This is incorrect because there might be more than one pair of bijections η and π and then $\psi \neq \pi\phi\eta^{-1}$. Following is a counter example.

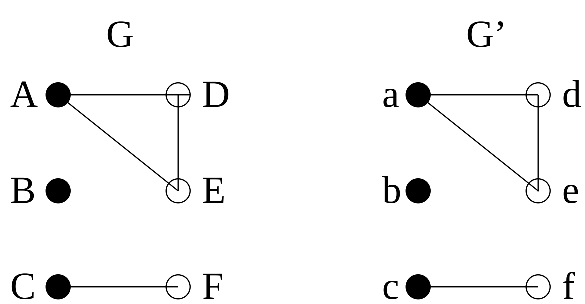


Figure 3: Colored graphs G and G' with two colors. Vertices $\{A, B, C, a, b, c\}$ and $\{D, E, F, d, e, f\}$ have the same color

Figure 4 shows the graph $f(G, G')$. It clearly does not have a set of vertices, one of each color from which no other vertex is reachable but the graphs G and G' are isomorphic which contradicts the main result of the paper. The graph does not satisfy Lemma 4 of the paper as well.

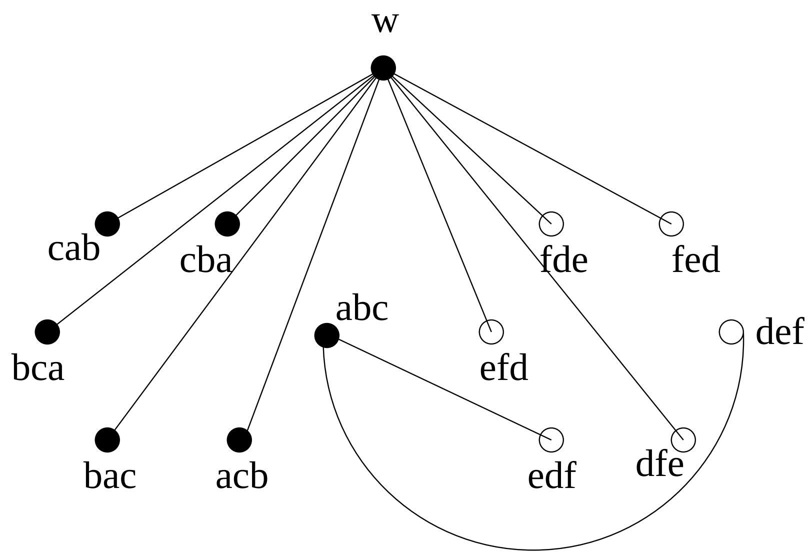


Figure 4: The graph $f(G, G')$ constructed according to the rules given in [1].

If we interpret **Rule 2** to say that it can be used for each pair of colors i, j and bijection ϕ exactly once, we would still end up constructing the same graph $f(G, G')$ as the rule would fire for colors 1 and 2 and bijection abc and also for colors 1 and 2 and bijection def.

References

[1] Birgit Jenner, Johannes Köbler, Pierre McKenzie, and Jacobo Torán. Completeness results for graph isomorphism. In *J. Comput. Syst. Sci.*, volume 66, pages 549–566, May 2003.

5 Appendix

In this appendix, we present an exhaustive case-analysis to find out all (α) and (β) constraints for a given pair of colored graphs and a given color. Let $G = (V, E, \{V_i\}_{i=1}^k)$ and $G' = (V', E', \{V_i'\}_{i=1}^k)$ be two colored graphs and $\phi : V_i \to V_i'$, $\psi : V_j \to V_j'$ be two bijections. From Lemma 2, $\phi \times \psi \in \mathrm{Iso}(G_{V_i \cup V_j}, G'_{V_i' \cup V_j'})$ if and only if $\phi \in \mathrm{Iso}(G_{V_i}, G'_{V_i'})$, $\psi \in \mathrm{Iso}(G_{V_j}, G'_{V_j'})$ and $\phi \times \psi \in \mathrm{Iso}(G_{V_i \cup V_j} - (E_{V_i} \cup E_{V_j}), G'_{V_i' \cup V_j'} - (E'_{V_i'} \cup E'_{V_j'}))$. We assume that $|V_i| = |V_j| = |V_i'| = |V_j'| = 3$. The cases for smaller size sets can be analyzed easily. Let $V_i = \{A, B, C\}$, $V_j = \{D, E, F\}$, $V_i' = \{a, b, c\}$ and $V_j' = \{d, e, f\}$.

We use $\phi = abc$ as a shorthand for $\phi = \{A \mapsto a, B \mapsto b, C \mapsto c\}$ and $\psi = def$ as a shorthand for $\psi = \{D \mapsto d, E \mapsto e, F \mapsto f\}$. Figure 5 shows the constraints generated for $\phi \in \mathrm{Iso}(G_{V_i}, G'_{V_i'})$ and Figure 6 shows the constraints generated for $\phi \times \psi \in \mathrm{Iso}(G_{V_i \cup V_j} - (E_{V_i} \cup E_{V_j}), G'_{V_i' \cup V_j'} - (E'_{V_i'} \cup E'_{V_j'}))$. As notational convenience, we use $\phi(A) = b$ to denote the set of (α) constraint in which ϕ must map A to b i.e. $\phi \neq abc$, $\phi \neq acb$, $\phi \neq cab$, $\phi \neq cba$. Also, we use $\beta(\pi_1, \pi_2)$ to denote the (β) constraint $\phi = \pi_1 \Leftrightarrow \psi = \pi_2$. To reduce the number of cases we must consider, we make use of the fact that for any pair of graphs H and H', $\mathrm{Iso}(H, H') = \mathrm{Iso}(\overline{H}, \overline{H}')$, where $\overline{H}$ and $\overline{H}'$ are complements of the respective graphs. This shows that in order to consider the isomorphisms in $\mathrm{Iso}(G_{V_i \cup V_j} - (E_{V_i} \cup E_{V_j}), G'_{V_i' \cup V_j'} - (E'_{V_i'} \cup E'_{V_j'}))$, we only need to consider graphs with at most 4 edges (cases with higher number of edges can be converted into cases with at most 4 edges by taking complements). Figures 5 and 6 only show the cases when it is possible to have an isomorphism.

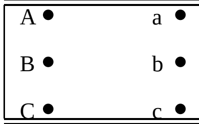
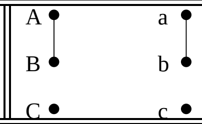
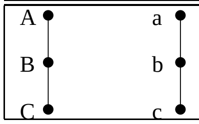
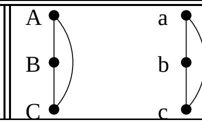
Case	Constraints	Case	Constraints
	$\emptyset$		$\phi(C) = c$
	$\phi(B) = b$		$\emptyset$

Figure 5: Constraints generated for deciding if $\phi \in \mathrm{Iso}(G_{V_i}, G'_{V'_i})$ where $V_i = \{A, B, C\}$ and $V'_i = \{a, b, c\}$.

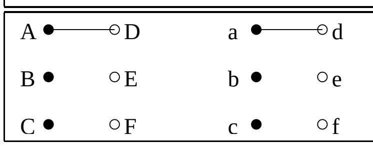
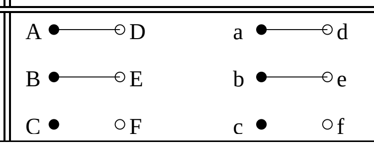
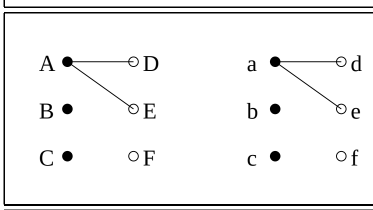
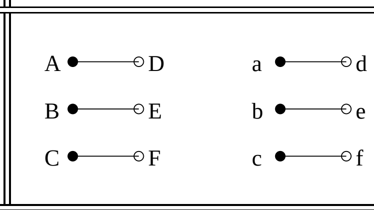
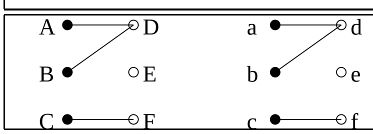
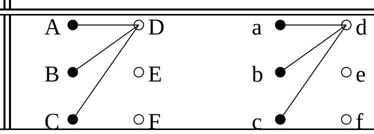
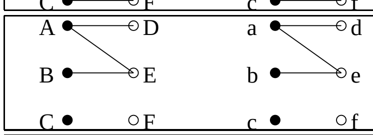
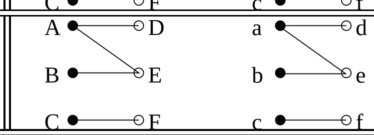
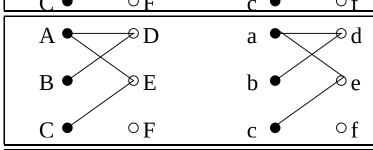
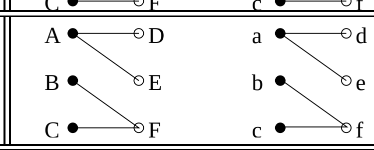
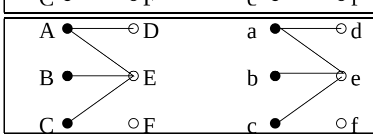
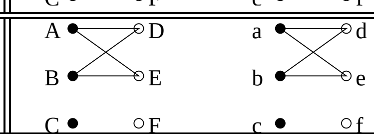
Case	Constraints	Case	Constraints
	$\phi(A) = a$ $\psi(D) = d$		$\phi(C) = c$ $\psi(F) = f$ $\beta(abc, def)$ $\beta(bac, edf)$
	$\phi(A) = a$ $\psi(F) = f$		$\beta(abc, def)$ $\beta(acb, dfe)$ $\beta(bac, edf)$ $\beta(bca, efd)$ $\beta(cab, fde)$ $\beta(cba, fed)$
	$\phi(C) = c$ $\psi = def$		$\psi(D) = d$
	$\phi = abc$ $\psi = def$		$\phi = abc$ $\psi = def$
	$\phi(A) = a$ $\psi(F) = f$ $\beta(abc, def)$ $\beta(acb, edf)$		$\phi(A) = a$ $\psi(F) = f$
	$\phi(A) = a$ $\psi = def$		$\phi(C) = c$ $\psi(F) = f$

Figure 6: Constraints generated for deciding if $\phi \times \psi \in \mathrm{Iso}(G_{V_i \cup V_j} - (E_{V_i} \cup E_{V_j}), G'_{V'_i \cup V'_j} - (E'_{V'_i} \cup E'_{V'_j}))$ where $V_i = \{A, B, C\}$, $V'_i = \{a, b, c\}$, $V_j = \{D, E, F\}$, $V'_j = \{d, e, f\}$.