CS412, Fall 19
Prof. Amos Ron

Assignment #6
Due: December 11, 2019
assignments handed in after Friday , 4 p.m will not be graded

buy-out policy for this assignment *Under certain conditions, you may buy-out one or two of the questions that appear in this assignment. The buyout's cost is 3 grace days per each question, and questions cannot be prorated (i.e., you cannot buy 2/3 of a question for two grace days.) Check carefully your grace days balance before you make a decision here.* **you must have turned in all the previous five assignments in order to be eligible for buyout**. *If you buyout a question, write its number, and write next to it:* **buy-out: three grace days**
Overdraft clause: If you do not have sufficiently many grace days, or had not turned in all previous assignments, your grade for that question will be 0. *Otherwise:*

(a) If you buyout one question, we will grade the other two questions and will assign you a grade for the assignment based solely on those other questions. Of course, if you do poorly on those questions, the buy-out may end in a fiasco.
(b) If you buyout two questions we will grade the remaining one. The rule is still as above.

(1) (This problem deals with numerical integration.) The midpoint rule is constructed to be exact for all constant polynomials. However, it is exact for all linear polynomials, too. Simpson's rule was designed to be exact for all quadratic polynomials.* However, it is exact for all cubics, too. The common property of these two rules is that, for both,

$$\int_a^b t^0 W(t)\,dt = \int_a^b W(t)\,dt = 0,$$

with $W(t) := (t-t_1)(t-t_2)...$, and with $t_1, t_2, \ldots$ the nodes of the scheme.

Rules that are even better can be constructed if one insists that $W(t)$ satisfies additional conditions such as

$$\int_a^b t^1 W(t)\,dt = 0, \quad \int_a^b t^2 W(t)\,dt = 0,$$

etc. The integral $\int_a^b t^j W(t)\,dt$ is called the "jth moment of the function w" (wrto the interval $[a,b]$). Thus, the name of the game is to choose the nodes in a way that w has as many zero moments as possible. If all the moments of w up to the jth are zero, your rule scores $j+1$ extra points, i.e., while you use n points in your rule (whatever n is), the error looks like you are using $n+j+1$ points. It is not hard to see that for a given n, the maximum possible value of j is $j = n-1$. In that case, you use n nodes, and the error looks like you are using $2n$ nodes. Such rules are called "Gaußian rules".

In this question you will construct a Gaußian rule for $n = 2$ and will observe the usefulness of it. (For $n = 1$ the Gaußian rule is midpoint.) Your general goal is to find an integration scheme of the form

$$\int_0^1 f(t)\,dt \approx w_1 f(t_1) + w_2 f(t_2). \tag{1}$$

* Simpson is exact for all quadratics since: if f is a quadratic polynomial, then any polynomial interpolant of f at three points (or more) coincides with f.

(a) Your choice for t_1, t_2 should be as follows. First, find a quadratic polynomial $W(t) = t^2 + bt + c$ that satisfies the following two conditions:

$$\int_0^1 W(t)\,dt = 0, \qquad \int_0^1 tW(t)\,dt = 0.$$

(Hint: plug in the expression you have for w into these integrals, and carry out the integration. You will get two equations with the two unknowns b, c.) Then find the two roots of $W(t)$. These are your t_1, t_2. (t_1 and t_2 are not rational numbers. If you wish to use an approximate decimal representation of them, be sure to use high precision approximation. If you found a closed form for those roots, e.g., $t_1 = \sqrt{2}/2$, keep it in this form).

(b) Now, find w_1 and w_2. (There are several ways how to find them. One way is to recall the general approach from class: once you are given t_1, t_2 and asked to derive a rule, you, at least theoretically, interpolate the underlying function by a (linear) polynomial at t_1 and t_2 and then integrate the polynomial. If you write that polynomial in the Lagrange form, the weights are then the integrals of the Lagrange polynomials. Check the notes for more details.)

(c) Your scheme is based on two nodes, hence clearly must be exact for all linears. However, if you did it right, you should be exact for all cubics (like a scheme that uses four points). Check that. Is it also correct for all quartics ('quartic'=degree 4)?

(d) Now apply your great scheme to $f(x) = e^x$, $f(x) = 1/(x+1)$, and $f(x) = \sin x$. Compare your accuracy with composite midpoint with two subintervals, and with (simple) Simpson. (You may use either `matlab` or a calculator here).

(2) In this question, you compare three methods for solving IVP's: (a) The Milne-Simpson predictor-corrector, (b) the Adams-Bashforth-Moulton method, and (c) the Runge-Kutta method of order 4. You will use each of the methods in order to approximate $y(b)$, with $y(t)$ the solution of an initial value problem

$$y' = f(t, y), \quad y(a) = y_0.$$

Note that Runge-Kutta uses four evaluations of f per each step, while the predictor-corrector methods use two evaluations per each step, hence once you choose some step-size h for Runge-Kutta, the 'fair comparison' should be with predictor-corrector with step size $h/2$ (why?).

(a) Write, for each of the above methods, a (small) routine that, given the function f, the numbers a, b, the initial value y_0, and a parameter h, applies the corresponding method for finding (approximately) $y(b)$. Note that in the predictor-corrector methods you need first to find y_1, y_2, y_3, before you can start the method, so use Runge-Kutta in order to find these three required values. Use the given h as the step-size for Runge-Kutta, and use step-size $h/2$ for each of the predictor-corrector methods. Test your routines by running them on some known example (e.g., some example from a text book).

(b) You are now given the IVP

$$y' = e^{-2t}y - y^2, \quad y(1) = .5, \quad y(2) = ?.$$

Apply each of your routines with $h = 1/4, 1/8 \ldots$ until the difference between two consecutive approximations (i.e., the one for the current h and the one for the previous h) is smaller then 10^{-6} (or until you determine that the method does not work). Repeat now with two other IVP of your choice (with respect to the same interval and the same initial value). Draw conclusions (concerning the relative performance of the three methods here).

(c) Use your calculations in (b) is order to determine the *order* of each method. (After you completed (b), you know more or less the correct value of $y(2)$, so you can calculate the errors committed when using $h = 1/4, 1/8, \ldots$. If you divide the error for a certain h by the error for the previous h you should be able to determine the order of your method).

(3) Your goal in this question is to practice solving IVP that involve several functions, as well as IVP of higher order (i.e., that involve derivatives of order 2 or higher.) You will do it in two steps: in the first step, you will observe that the methods that we discussed in class for solving a single equation apply as well to the system case. In the second step you will realize that a second order differential equation can be reduced to a system of two first order equations. You can either use `matlab` or a calculator for the sake of the computations.

(a) Show that the functions

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} t - 2/t \\ 2 \ln t \end{pmatrix}$$

satisfy the initial value problem:

$$y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 + 2e^{-y_2} \\ t - y_1 \end{pmatrix}, \quad y(1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Then (as a warm up) apply one step of Euler's method (with $h = .1$) in order to approximate the solution at 1.1. Compare with the exact solution.

(b) Now, use the Runge-Kutta (RK4) method for the same purpose (i.e., approximating the solution at 1.1). First, do that in one step ($h = .1$), and then try two steps ($h = .05$). Find the ratio between the two errors you obtained here. Any conclusions?

(c) After doing (a),(b) you have a basic idea how systems are solved. Suppose now that instead of a system of two equations you have a single equation of order 2, e.g., we look for a function $y(t)$ such that

$$y'' = y'e^y + ty, \quad y(1) = 1, \; y'(1) = 0.5 \tag{1}$$

(Note that now we specify $y'(a)$ as well in the IVP, not only $y(a)$). Define a vector $x(t)$ of two functions: $x_1(t) := y(t)$, and $x_2(t) := y'(t)$. Now, you need to convert the second order IVP for y to a first order IVP on x (Hint: look at the notes, or wait until we cover it in class.)

(d) At this point, you can solve numerically the original IVP for y, by solving the equivalent IVP on x. Apply one step of predictor-corrector with Euler the predictor and trapezoid the corrector in order to find $y(1.1)$ (i.e., $h = .1$.)

(e) Find the solution y on a dense mesh on the interval $[1, 2]$ and plot its graph. (Note: you can use any method, but you will need to know that your solution is reasonably accurate.)