1 Review

For the last few lectures, we have discussed the solution of initial value problems for first order differential equations. Recall that the initial value problem is a problem of the form:

**Problem 1.1.** Given

\[
\begin{align*}
y'(t) & = f(t, y(t)) \\
y(a) & = y_0
\end{align*}
\]

find \(y(b)\).

We have discussed several methods for solving the IVP, all using the idea of splitting the interval \([a, b]\) into \(N\) subintervals using partition points \((t_0, t_1, \ldots, t_N)\), then approximating the definite integral of \(y'\) over each subinterval. In this lecture, we shall discuss the solution of higher order differential equations.

2 Higher Order Differential Equations

To begin, we consider second order differential equations. When solving second order differential equations we would like to find a function \(y(t)\) given an expression for its second derivative in terms of \(t\), \(y(t)\), and \(y'(t)\):

**Problem 2.1.** Given the second order differential equation:

\[
y''(t) = f(t, y(t), y'(t))
\]

find \(y(t)\).

As was the case with first order differential equations, this is not a well-defined problem. For example, consider the second order differential equation:

\[
y''(t) = -y(t)
\]

It is not difficult to see that functions like:

\[
\begin{align*}
y(t) & = k_1 \cdot \sin(t) \\
y(t) & = k_2 \cdot \cos(t)
\end{align*}
\]
are correct solutions. In fact, we can characterize all correct solutions using the linear combination.

\[ y(t) = k_1 \cdot \sin(t) + k_2 \cdot \cos(t) \]

Note that this general solution has two parameters, and therefore we need two pieces of information to formulate the problem so that there is a unique solution. We will consider two correct formulations of the problem:

**Initial Value Problem**

In an initial value problem, we are given the values of \( y \) and \( y' \) at the same point \( a \) and want to find the value of \( y \) at a second point \( b \).

**Problem 2.2.** Given:

\[
\begin{align*}
y''(t) &= f(t, y(t), y'(t)) \\
y(a) &= y_0 \\
y'(a) &= y'_0
\end{align*}
\]

find \( y(b) \).

**Boundary Value Problem**

In a boundary value problem, we are given the value of \( y \) at two different points \( a \) and \( b \), and want to find the value of \( y \) at some third point \( c \), which is usually in the interval \( [a, b] \).

**Problem 2.3.** Given:

\[
\begin{align*}
y''(t) &= f(t, y(t), y'(t)) \\
y(a) &= y_0 \\
y(b) &= y_1
\end{align*}
\]

find \( y(c) \).

Solving boundary value problems is far more complex than solving initial value problems. We shall focus on initial value problems for the next two lectures.

(One could wonder: Does it make the problem well-posed to give the second derivative at \( a \) as our third piece of information? In other words, is the problem “Given

\[
\begin{align*}
y''(t) &= f(t, y(t), y'(t)) \\
y(a) &= y_0 \\
y''(a) &= y''_0
\end{align*}
\]

find \( y(c) \)” well-posed? The answer is no. Either \( y''(a) \) is redundant or it is contradictory. In our example above, suppose \( y(0) = 1 \). Then \( y''(0) = -1 \) is redundant with \( y''(t) = -y(t) \), while \( y''(0) = 2 \) contradicts \( y''(t) = -y(t) \). Similarly, giving \( y' \) (a point different from \( a \)) doesn’t work.)
3 Solution of Second Order IVP

We use substantially the same method for solving a second order initial value problem that we used for solving a first order initial value problem. The trick is to reduce the second order IVP to a system of first order IVPs, which we then solve in tandem. Given a second order IVP:

**Problem 3.1.** Given:

\[
\begin{align*}
  y''(t) & = f(t, y(t), y'(t)) \\
  y(a) & = y_0 \\
  y'(a) & = y'_0
\end{align*}
\]

find \( y(b) \).

we first define two new functions \( x_1(t) \) and \( x_2(t) \):

\[
\begin{align*}
  x_1(t) & = y(t) \\
  x_2(t) & = y'(t) = x'_1(t)
\end{align*}
\]

We can then rewrite the second order differential equation from problem 3.1 as two first order differential equations:

\[
\begin{align*}
  x'_1(t) & = x_2(t) \\
  x'_2(t) & = f(t, x_1(t), x_2(t))
\end{align*}
\]

Then we rewrite the initial values for \( y \) and \( y' \) in terms of \( x_1 \) and \( x_2 \):

\[
\begin{align*}
  x_1(a) & = y_0 \\
  x_2(a) & = y'_0
\end{align*}
\]

As before, we solve the problem by splitting up the interval \([a, b]\) into \( N \) equal-sized partitions using partition points \((t_0, t_1, \ldots, t_N)\). We must approximate both functions on each subinterval, so we calculate two approximate values at each point \( t_i \): \( X_{1,i} \) and \( X_{2,i} \), then use \( f \) to calculate \( X'_{2,i} \). Note that \( X'_{1,i} = X_{2,i} \) so that we only calculate three total values at each point. Here we define \( X_{i,j} \) as our approximation of \( x_i(t_j) \) and \( X'_{i,j} \) as our approximation of \( x'_i(t_j) \).

Now we can apply any method that we have studied for solving first order initial value problems to solve this problem. The only issue is that we must calculate all the approximations \( X_{1,i}, X_{2,i}, X'_{2,i} \) at point \( t_i \) before we can calculate the approximations at the next point \( t_{i+1} \), since the values are coupled (we use the \( X_2 \) values to approximate \( X_1 \), and we need both \( X_1 \) and \( X_2 \) values to calculate \( X'_{1} \)). To illustrate, consider the following example

**Example 3.1.** Given the second order IVP:

\[
\begin{align*}
  y''(t) & = -ty(t) + y'(t)^2 \\
  y(2) & = 5 \\
  y'(2) & = -1
\end{align*}
\]

find \( y(3) \).
Solution:

1. Define functions $x_1(t)$, $x_2(t)$:

   \[
   x_1(t) = y(t) \\
   x_2(t) = y'(t)
   \]

2. Rewrite as a system of first order differential equations

   \[
   x_1'(t) = x_2(t) \\
   x_2'(t) = -t \cdot x_1(t) + x_2(t)^2
   \]

3. Define initial values for the first partition point $t_0$:

   \[
   X_{1,0} = 5 \\
   X_{2,0} = -1 \\
   X_{2,0}' = -t \cdot X_{1,0} + X_{2,0}^2 \\
   \text{= } -0(5) + (-1)^2 \\
   \text{= } 1
   \]

4. Solve using any method for first order IVP

4 Systems of First Order IVP

Note that this method for solving second order IVP is a special case of a general method for solving systems of first order IVP. Given the system of first order initial value problems:

\[
\begin{align*}
    x_1'(t) &= f_1(t, x_1(t), x_2(t)) \\
    x_2'(t) &= f_2(t, x_1(t), x_2(t)) \\
    x_1(a) &= x_{1,0} \\
    x_2(a) &= x_{2,0}
\end{align*}
\]

We can solve this system by calculating our approximations for the value of each function $x_i$ at each partition point $t_j$ together (i.e. $X_{i,j}$).

4.1 Adapting Euler’s Method to Systems of First Order Differential Equations

Recall that, to find the approximation $Y_{j+1}$ of $y(t_{j+1})$ with Euler’s method, we use:

\[
Y_{j+1} = Y_j + h \cdot Y'_j
\]
To adapt this rule to systems of first order differential equations, we must calculate the two approximations $X_{1,j+1}, X_{2,j+1}$:

\[
X_{1,j+1} = X_{1,j} + h \cdot X'_{1,j} \\
X_{2,j+1} = X_{2,j} + h \cdot X'_{2,j}
\]

**Example.**
Say, $h = .1, t_0 = 0$,

\[
X_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

and

\[
X'_{1,0} = 1 \\
X'_{2,0} = 1
\]

so

\[
X'_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

We use Euler:

\[
X_{1,1} = X_{1,0} + hX'_{1,0} \\
= 0 + .1 \cdot 1 = .1 \\
X_{2,1} = X_{2,0} + hX'_{2,0} \\
= 1 + .1 \cdot 0 = 1
\]

so

\[
X_1 = \begin{bmatrix} .1 \\ 1 \end{bmatrix}.
\]