CS412, Spring 04
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RK4

The Runge-Kutta method of order 4 was by far the most popular IVP solver. It nowadays competes with the ABM predictor-corrector. Predictor-corrector methods have the extra benefit that they provide *two* different approximate values to $y(t_j)$ (the predicted one and the corrected one). One can compare the two values and determine based on those whether the error committed in the step seems to be too large. In this way, people may adjust their step-size h locally. This methodology leads to the so-called *adaptive algorithms*, where the step size is modified on the run. It is not hard to understand the qualitative underpinning of this idea: denoting the approximate values provided by the predictor and the corrector by y_{jp} and y_{jc} (respectively), a large difference $y_{jp} - y_{jc}$ is an indication that the error is quite large (for example if the corrector reduced the error of the predictor by a factor of ≈ 5, then the above difference is about 4 or 6 times larger than the error of y_{jc} (why?)). In fact, arguments like the above are used in order to derive from the pair (y_{jc}, y_{jp}) a third value which is even better (a la extrapolation idea that we learned elsewhere).

Going back to RK4, it attempts to do Simpson on the interval $[t_j, t_{j+1}]$. Since the function we integrate is y', the method, ideally, would look like

$$y_{j+1} := y(t_j) + \frac{h}{6}(y'(t_j) + 4y'(t_j + \frac{h}{2}) + y'(t_{j+1})).$$

Since we know none of the values on the right hand side, we must replace them by some approximate values. $y(t_j)$ is trivially replaced by y_j. $y'(t_j)$ is trivially replaced by $A_1 := f(t_j, y_j)$ (recall that $y'(t_j) = f(t_j, y(t_j))$). It is not hard to find reasonable approximations to the remaining two values. However, if you would like the local error to behave like the local error of Simpson (viz., $O(h^5)$), you must take great precaution here. You *cannot* find, e.g., $y'(t_{j+1})$ to the requisite accuracy (at least not easily). The RK4 method is designed such that the (relatively large) errors that are produced in the approximations to $y'(t_j + h/2)$ and $y'(t_{j+1})$ cancel to a large degree each other. (This is yet another form of extrapolation).

The details are as follows. The definition is

$$y_{j+1} := y_j + \frac{h}{6}(A_1 + 2A_2 + 2A_3 + A_4).$$

A_1 is as above. A_2 and A_3 are two different approximations to $y'(t_j + h/2) = f(t_j + \frac{h}{2}, y(t_j + \frac{h}{2}))$, and A_4 is an approximation to $y'(t_{j+1}) = f(t_{j+1}, y(t_{j+1}))$.

Step I: finding A_2 and A_3. We just need to approximate $y(t_j + \frac{h}{2})$). A tangent line approximation at t_j gives the value

$$y(t_j + \frac{h}{2}) \approx y(t_j) + .5hy'(t_j) \approx y_j + .5hA_1. \tag{1}$$

Plugging that value into the DE, we get

$$A_2 := f(t_j + \frac{h}{2}, y_j + .5hA_1).$$

Once we found A_2, we may get a fresh approximation to $y(t_j + \frac{h}{2})$:

$$y(t_j + \frac{h}{2}) \approx y(t_j) + .5hy'(t_j + \frac{h}{2}) \approx y_j + .5hA_2. \tag{2}$$

This gives us another approximation of $y'(t_j + \frac{h}{2})$, namely

$$A_3 := f(t_j + \frac{h}{2}, y_j + .5hA_2).$$

Note that A_2 and A_3 are both biased. Both attempt to march from t_j to $t_j + \frac{h}{2}$ based on derivative info. One uses the derivative at t_j the other uses the derivative at $t_j + \frac{h}{2}$. In fact, using the derivative half the way (i.e., at $t_j + \frac{h}{4}$) would have been better. This heuristic explains why we average A_2 and A_3 (rather than taking A_3).

Step II: Finally, a good approximation for $y(t_{j+1})$ is given by

$$y(t_{j+1}) \approx y(t_j) + hy'(t_j + \frac{h}{2}) \approx y_j + hA_3.$$

Thus,

$$A_4 := f(t_{j+1}, y_j + hA_3).$$

I wrote all the above without looking at any text book (but, needless to say, have just checked to see that the details are correct). The moral is that scientific algorithms, even hairy ones, are sometimes based on simple heuristics, and can be recovered to their fullest details if you understand the heuristics.

The good news is that despite of the content of the previous paragraph, you will not be expected to remember the details of RK4 for the final exam (the same applies to AB and AM. I do expect you to remember the details of every other IVP method).