CS515 Spring 08
Prof. Ron

Assignment #2

Due February 5, 2008

Question # 1.
Check whether the following maps are linear functionals (below, $t_0, t_1 \in \mathbb{R}$ are fixed.)
(a) $\lambda : f \mapsto f(t_0)$.
(b) $\lambda : f \mapsto f''(t_0)$.
(c) $\lambda : f \mapsto f(t_0) - f'(t_1)$.
(d) $\lambda : f \mapsto f(t_0)f(t_1)$.
(e) $\lambda : f \mapsto \int_{t_0}^{t_1} |f(t)|dt$.
(f) $\lambda : f \mapsto \int_{t_0}^{t_1} f(t)t^2 dt$.
(g) $\lambda : f \mapsto \max\{f(t) : \ 0 \le t \le 1\}$.

Question # 2.
Let

$$B_1(t) := \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $B_2 := B_1 * B_1$, and $B_3 := B_1 * B_2$. Draw the graph of (the interesting part of) B_2 and of B_3. (B_2 is called a 'hat function', while B_3 is known as a 'quadratic B-spline'.)

Question # 3.
Prove that, for functions f and g (defined on $\mathbb{R}$),

$$f * g = g * f.$$

Question # 4.
Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined by

$$f(t) := t.$$

(a) Compute the Fourier coefficients of f.
(b) The function f seems to be a nice smooth function. On the other hand, its Fourier coefficients decay very slowly. Can you settle this apparent paradox?

Question # 5.
Let $H : \mathbb{R} \to \mathbb{R}$ be the *Haar function*:

$$H(t) := \begin{cases} 1, & 0 \le t < 1/2, \\ -1, & 1/2 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$H_{1,0} : t \mapsto H(2t), \quad H_{1,1} : t \mapsto H(2t-1), \quad H_{1,-1} : t \mapsto H(2t+1).$$

(a) Draw the graphs of the above four functions.
(b) Show that $(H, H_{1,0}, H_{1,1}, H_{1,-1})$ is an *orthogonal sequence* in the sense that

$$\langle f, g \rangle = 0$$

for any two different f, g from that sequence.
(c) Normalize each function in the above sequence to make the sequence *orthonormal*.