Question # 1.
Check whether the following maps are linear functionals (below, \( t_0, t_1 \in \mathbb{R} \) are fixed.)

(a) \( \lambda : f \mapsto f(t_0) \).
(b) \( \lambda : f \mapsto f''(t_0) \).
(c) \( \lambda : f \mapsto f(t_0) - f'(t_1) \).
(d) \( \lambda : f \mapsto f(t_0)f(t_1) \).
(e) \( \lambda : f \mapsto \int_{t_0}^{t_1} |f(t)|\,dt \).
(f) \( \lambda : f \mapsto \int_{t_0}^{t_1} f(t)t^2 \,dt \).
(g) \( \lambda : f \mapsto \max\{f(t) : 0 \leq t \leq 1\} \).

Question # 2.
Let \( B_1(t) := \begin{cases} 
1, & 0 \leq t < 1, \\
0, & \text{otherwise.}
\end{cases} \)

Compute \( B_2 := B_1 \ast B_1 \), and \( B_3 := B_1 \ast B_2 \). Draw the graph of (the interesting part of) \( B_2 \) and of \( B_3 \). (\( B_2 \) is called a ‘hat function’, while \( B_3 \) is known as a ‘quadratic B-spline’.)

Question # 3.
Prove that, for functions \( f \) and \( g \) (defined on \( \mathbb{R} \)),
\[
f \ast g = g \ast f.
\]

Question # 4.
Let \( f : [-\pi, \pi] \to \mathbb{R} \) be defined by \( f(t) := t \).

(a) Compute the Fourier coefficients of \( f \).
(b) The function \( f \) seems to be a nice smooth function. On the other hand, its Fourier coefficients decay very slowly. Can you settle this apparent paradox?

Question # 5.
Let \( H : \mathbb{R} \to \mathbb{R} \) be the Haar function:
\[
H(t) := \begin{cases} 
1, & 0 \leq t < 1/2, \\
-1, & 1/2 \leq t < 1, \\
0, & \text{otherwise.}
\end{cases}
\]

Let \( H_{1,0} : t \mapsto H(2t), \ H_{1,1} : t \mapsto H(2t - 1), \ H_{1,-1} : t \mapsto H(2t + 1) \).

(a) Draw the graphs of the above four functions.
(b) Show that \((H, H_{1,0}, H_{1,1}, H_{1,-1})\) is an orthogonal sequence in the sense that
\[
\langle f, g \rangle = 0
\]
for any two different \( f, g \) from that sequence.
(c) Normalize each function in the above sequence to make the sequence orthonormal.