1. To show that $\lambda$ is a linear functional we need to check additivity and homogeneity.

(a) $\lambda$ is a linear functional:

$$\lambda(f + g) = (f + g)(t_0) = f(t_0) + g(t_0) = \lambda(f) + \lambda(g)$$

$$\lambda(cf) = (cf)(t_0) = c f(t_0) = c \lambda(f)$$

(b) $\lambda$ is a linear functional:

$$\lambda(f + g) = (f + g)'(t_0) = f'(t_0) + g'(t_0) = \lambda(f) + \lambda(g)$$

$$\lambda(cf) = (cf)'(t_0) = c f'(t_0) = c \lambda(f)$$

(c) $\lambda$ is a linear functional:

$$\lambda(f + g) = (f + g)(t_0) - (f + g)'(t_1) = f(t_0) - f'(t_1) + g(t_0) - g'(t_1) = \lambda(f) + \lambda(g)$$

$$\lambda(cf) = (cf)(t_0) - (cf)'(t_1) = c f(t_0) - c f'(t_1) = c(f(t_0) - f'(t_1)) = c \lambda(f)$$

(d) $\lambda$ is not a linear functional.

Counterexample:

$$\lambda(2f) = (2f)(t_0) (2f)(t_1) = 4f(t_0)f(t_1) \neq 2\lambda(f).$$

(e) $\lambda$ is not a linear functional (except when $t_0 = t_1$).

Counterexample: Choose $f(t) \equiv 1$, $g(t) \equiv -1$. Then

$$\lambda(f + g) = \int_{t_0}^{t_1} |1 - 1|dt = 0$$

$$\lambda(f) + \lambda(g) = \int_{t_0}^{t_1} |1|dt + \int_{t_0}^{t_1} |-1|dt = 2|t_1 - t_0|.$$
Then
\[
\lambda(f + g) = 0 \neq 1 = 1 + 0 = \lambda(f) + \lambda(g)
\]

2.

\[
B_2(t) = \int_{\mathbb{R}} B_1(u) B_1(t-u) du = \int_0^1 B_1(t-u) du
\]

\[
= \begin{cases} 
0, & t \leq 0 \ (t-u \notin [0,1]), \\
\int_0^t du, & 0 < t \leq 1 \ (\text{to have } 0 < t-u \text{ we need } u < t), \\
\int_{t-1}^1 du, & 1 < t \leq 2 \ (\text{to have } t-u < 1 \text{ we need } u > t-1), \\
0, & 2 < t \ (t-u \notin [0,1]),
\end{cases}
\]

\[
= \begin{cases} 
0, & t \leq 0, \\
t, & 0 < t \leq 1, \\
2-t, & 1 < t \leq 2, \\
0, & 2 < t.
\end{cases}
\]

\[
B_3(t) = \int_{\mathbb{R}} B_2(u) B_1(t-u) du = \int_0^1 u B_1(t-u) du + \int_1^2 (2-u) B_1(t-u) du
\]

\[
= \begin{cases} 
0, & t \leq 0, \\
\int_0^t u du, & 0 < t \leq 1, \\
\int_1^t u du + \int_1^t (2-u) du, & 1 < t \leq 2, \\
\int_{t-1}^2 (2-u) du, & 2 < t \leq 3, \\
0, & 3 \leq t,
\end{cases}
\]

\[
= \begin{cases} 
0, & t \leq 0, \\
\frac{t^2}{2}, & 0 < t \leq 1, \\
-t^2 + 3t - \frac{3}{2}, & 1 < t \leq 2, \\
\frac{(3-t)^2}{2}, & 2 < t \leq 3, \\
0, & 3 < t.
\end{cases}
\]

3.

\[
f \ast g(t) := \int_{-\infty}^{\infty} f(u) g(t-u) du.
\]
Make a substitution \( s = t - u \). Then
\[
 f * g(t) = -\int_{-\infty}^{\infty} f(t-s)g(s)ds = \int_{-\infty}^{\infty} g(s)f(t-s)ds = g * f(t). 
\]

4. (a) Fourier coefficients \( \hat{f}(k) \) of \( f \) are
\[
\hat{f}(k) = \int_{-\pi}^{\pi} te^{-ikt}dt = \frac{2\pi i \cos k\pi}{k}.
\]
The Fourier coefficient at \( k = 0 \) should be calculated separately, and can be shown to be 0.
(b) The function \( f \) is not continuous as a periodic function, therefore its Fourier coefficients decay slowly.

5. (a)\( H, H_{1,0}, H_{1,1}, \) and \( H_{1,-1} \)

(b) \( H_{1,-1}, H_{1,0}, \) and \( H_{1,1} \) are mutually orthogonal because they have disjoint supports. For the same reason, \( H_{1,-1} \) is orthogonal to \( H \). Now,
\[
\langle H, H_{1,0} \rangle = \int_{\mathbb{R}} H(t)\overline{H_{1,0}(t)}dt = \int_{0}^{1/4} dt + \int_{1/4}^{1/2} (-1)dt = 0,
\]
and
\[ \langle H, H_{1,1} \rangle = \int_{\mathbb{R}} H(t)H_{1,1}(t) dt = \int_{3/4}^{1} dt + \int_{1/2}^{3/4} (-1) dt = 0. \]

(c) \[ \langle H, H \rangle = \int_{0}^{1} dt = 1, \]
so \( H \) does not require normalization.

\[ \langle H_{1,0}, H_{1,0} \rangle = \int_{0}^{1/2} dt = 1/2, \]
so \( H_{1,0} \) has to be multiplied by \( \sqrt{2} \) in order to be normalized. Since \( H_{1,-1} \) and \( H_{1,1} \) are shifts of \( H_{1,0} \), they have to be normalized by the same factor.