CS515 Spring 08
Prof. Ron

Assignment #3

Due February 12, 2008

Question # 1.
Show that, for $a \neq 0$, the dilation operator

$$(\mathcal{D}_a)f : t \mapsto \sqrt{a}\, f(at)$$

is *unitary*, i.e., it preserves inner products in the sense that

$$\langle f, g\rangle = \langle \mathcal{D}_a f, \mathcal{D}_a g\rangle, \quad \forall f, g \in L_2.$$

Question # 2.
Let the Haar function $H_0 := H$ be as in the previous assignment, and set $H_1 := \mathcal{D}_2 H$, and $H_2 = \mathcal{D}_4 H$. Check first whether the Haar function is orthogonal to all linear polynomials.

Now, create three functions f_1, f_2, f_3 as follows: f_1 have jump discontinuity at $t = .1$ (take f_1 to be a suitable translateof B_1), f_2 is continuous everywhere, but is not differentiable at $t = .1$ (a translate of B_2 will do), and f_3 is some terrific function (e.g., $f_3(t) = e^t$). Now, compute all the inner products

$$\langle f_j, H_k\rangle, \quad j = 1, 2, 3, \quad k = 0, 1, 2.$$

(there are nine products to compute altogether). Based on the all the above, write a short paragraph on the topic "the inability of the Haar function to see only roughness in functions".

Question # 3.
Show that the Haar system $(H_{j,k} : j, k \in \mathbb{Z})$ is *orthonormal*, i.e., the norm of each $H_{j,k}$ is 1, and

$$\langle H_{j,k}, H_{j',k'}\rangle = 0, \quad \text{unless } j = j' \text{ and } k = k'.$$

Here, for a given function ψ, the notation $\psi_{j,k}$ stands for the function

$$\psi_{j,k} : t \mapsto \sqrt{2^j}\, \psi(2^j t + k),$$

Hint: use the result from the previous question to simplify your argument. You might also wish to look at Assignment 2.

Question # 4.
Show that the Shannon system $(S_{j,k} : j, k \in \mathbb{Z})$ is orthonormal. To recall, the Shannon wavelet S is the function whose Fourier transform is

$$\widehat{S}(\omega) := \begin{cases} 1, & \pi \leq |\omega| \leq 2\pi, \\ 0, & \text{otherwise.} \end{cases}$$

As before,

$$S_{j,k} : t \mapsto \sqrt{2^j}\, S(2^j t + k),$$

Hint: there is no need to compute S (although you are asked to do so in the next question). Use Parseval's identity instead. Again, the result from the first question in this assignment is helpful here.