Question # 1.
Let \( F = \Pi_3 = \text{ran} W \), with \( W = \{0, 1, 2, 3\} \) (hence \( (W a)(x) = a(0) + a(1)x + a(2)x^2 + a(3)x^3 \)); also let \( \Lambda' : f \mapsto (f(0), f'(0), f(1), f'(1)) \).

(i) Prove that \( \Lambda' \) is 1-1 on \( \Pi_3 \). (Hint: how many zeros, counting multiplicities, must a polynomial \( f \) have if \( \Lambda' f = 0 \)?)
(ii) Construct the Gram matrix \( \Lambda' W \).
(iii) Construct the basis \( V \) for \( \Pi_3 \) for which \( \Lambda' V = \text{id} \).
(iv) Construct the unique cubic polynomial that matches the function \( x \mapsto \sin(\pi (x - 1/2)) \) in value and slope at the two points, 0 and 1.

Question # 2.
The MATLAB Spline Toolbox command \( \text{bs} = \text{spmak}(t, 1) \); returns a description of the B-spline with knot sequence \( t(1:end) \) whose values at the entries of the vector (or matrix) \( x \) can be obtained by the command \( \text{fnval(bs,x)} \). (Note that MATLAB will determine the order \( k \) of the B-spline from the length of the vector \( t \).)

Generate (and hand in, along with a listing of the MATLAB script that generated it) a plot that shows, for \( t := (t_1, \ldots, t_4) = (0, 2, 3, 4, 7) \) and on the interval \([-1..6]\), (i) \( B_{1,3,t} \); (ii) \( \omega_{1,3}B_{1,2,t} \); (iii) \( (1 - \omega_{2,3})B_{2,2,t}, \) with, to recall, \( \omega_{i,k}(t) = (t - t_i)/(t_{i+k-1} - t_i) \).

Comment on the significance of this plot.

Comment on notation: the B-spline \( B_{j,k,t} \) is the B-spline \( B_{j,k} \) with respect to the knot sequence \( t = (t_0, \ldots, t_N) \).

Question # 3.
Use the recurrence relation to construct

\[
B(\cdot|0, 1, 1), \quad B(\cdot|0, 0, 1);
B(\cdot|0, 1, 1, 1), \quad B(\cdot|0, 0, 1, 1), \quad B(\cdot|0, 0, 0, 1);
B(\cdot|0, 1, 1, 1, 1), \quad B(\cdot|0, 0, 1, 1, 1), \quad B(\cdot|0, 0, 0, 1, 1), \quad B(\cdot|0, 0, 0, 0, 1).
\]

Can you guess a formula for

\[
B(\cdot|0, \ldots, 0, 1, \ldots, 1) \quad \text{with } i \text{ terms and } j \text{ terms}
\]

Question # 4.
Use the de Boor - Fix dual functionals \( \lambda_{jk} \) in order to prove the following property of splines: if \( f \) is a linear combination of B-splines, and it vanishes outside an interval \([t_{j+1}, t_{j+k}]\) for some \( j \) (and where \( k \) is the order of the B-splines), then \( f \) is zero everywhere.

Note that this last result is tight: a spline \( f \) that vanishes outside an interval of the form \([t_j, t_{j+k}]\) needs not to be 0 everywhere (why?)