(1) Let \((B_j)_{j=-\infty}^{\infty}\) be the cubic B-spline basis associated with the knot sequence \(t := h\mathbb{Z}\), with \(h\) some (small) positive number. Your goal is to construct an approximation scheme

\[ f \mapsto Af := \sum_{j=-\infty}^{\infty} \mu_j(f)B_j, \]

such that \(A\) is the identity on cubic polynomials, and, for each \(j\)

\[ \mu_j(f) = af(h(j + 1)) + bf(h(j + 2)) + cf(h(j + 3)) \]

(i.e., \(\mu_j\) involves evaluation of \(f\) at the three interior knots of the B-spline \(B_j\), and nowhere else).

1. Find the value of the coefficients \(a, b, c\). (Hints: those values do not depend on \(j\), since the mesh is uniform. Choose a basis for the cubic polynomials, and find first the values the de Boor-Fix functionals attain on the cubic polynomial basis. Your \(\mu_j\)'s should attain exactly the same values).

2. Employ your scheme in exactly the same setup as in Assignment 7 (81 knots on the interval \([-2, 2]\) etc.) and compare the error of your scheme to the errors of the other schemes from Assignment 7.

(2) Let \(\phi\) be the function

\[ \phi(t) := \begin{cases} 1, & 0 \leq t < 1, \\ 0, & \text{otherwise.} \end{cases} \]

(a) Prove that the function is refinable, and find its \(2\pi\)-periodic refinement mask \(H_0(\omega)\).

(b) What exactly is the transfer operator \(T\) associated with \(\phi\)?

(c) Show that \(T1 = 1\) (here, ‘1’ is the \(2\pi\)-periodic constant function).

(d) What property of the shifts \(E(\phi)\) of \(\phi\) is connected to the statement in (c)? State a relevant theorem studied in class.

(3)

(a) Define B-splines.

(b) Use your definition to conclude that \(B(\cdot | -1, 1, 2)\) is positive on the interval \((-1 \ldots 2)\) and is zero elsewhere.
(4) Consider the linear map $\Lambda' : g \mapsto (g(\tau_1), \ldots, g(\tau_n))$ with $\tau_1 < \cdots < \tau_n$ in some interval $I$.

(a) State the Schoenberg-Whitney Theorem.

(b) Choose a knot sequence $t$ so that $\Lambda'_t$ maps $S_{3,t}$ 1-1 onto $\mathbb{R}^n$ (explain).

(c) What practical purpose could the resulting spline space $S_{3,t}$ serve?

(5) Let $X = \mathbb{R}^I$ be the linear space of all real-valued functions on the interval $I := [0 . . 4]$, and consider the linear map $V : \mathbb{R}^3 \to X : a \mapsto \sum_{i=1}^{3} v_i a(i)$ with

$$v_1(s) := (s - 2)(s - 3), \quad v_2(s) := (s - 3)(s - 1), \quad v_3(s) := (s - 1)(s - 2).$$

(a) Prove that $V$ is 1-1.

(b) Prove that $\text{ran} V = \Pi_2$.

(c) Construct a linear map $\Lambda' : X \to \mathbb{R}^3$ for which $\Lambda' V = \text{id}$.

(note:) You may prefer to reorder this list of requests.

(d) Which of these maps is an analysis operator, a synthesis operator, a reconstruction map, a data map? Explain.