CS515 Spring 08
Prof. Ron

Final Exam

due May 09, 2008

(1) Let $(B_j)_{j=-\infty}^{\infty}$ be the cubic B-spline basis associated with the knot sequence $\underline{t} := h\mathbb{Z}$, with h some (small) positive number. Your goal is to construct an approximation scheme

$$f \mapsto Af := \sum_{j=-\infty}^{\infty} \mu_j(f) B_j,$$

such that A is the identity on cubic polynomials, and, for each j

$$\mu_j(f) = af(h(j+1)) + bf(h(j+2)) + cf(h(j+3))$$

(i.e., μ_j involves evaluation of f at the three interior knots of the B-spline B_j, and nowhere else).

(1) Find the value of the coefficients a, b, c. (Hints: those values do not depend on j, since the mesh is uniform. Choose a basis for the cubic polynomials, and find first the values the de Boor-Fix functionals attain on the cubic polynomial basis. Your μ_j's should attain exactly the same values).

(2) Employ your scheme in exactly the same setup as in Assignment 7 (81 knots on the interval $[-2, 2]$ etc.) and compare the error of your scheme to the errors of the other schemes from Assignment 7.

(2) Let ϕ be the function

$$\phi(t) := \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Prove that the function is refinable, and find its 2π-periodic refinement mask $H_0(\omega)$.

(b) What exactly is *the transfer operator* T *associated with* ϕ?

(c) Show that $T1 = 1$ (here, '1' is the 2π-periodic constant function).

(d) What property of the shifts $E(\phi)$ of ϕ is connected to the statement in (c)? State a relevant theorem studied in class.

(3)

(a) Define B-splines.

(b) Use your definition to conclude that $B(\cdot| -1, 1, 2)$ is positive on the interval $(-1\,..\,2)$ and is zero elsewhere.

(4) Consider the linear map $\Lambda'_\tau : g \mapsto (g(\tau_1), \ldots, g(\tau_n))$ with $\tau_1 < \cdots < \tau_n$ in some interval I.

(a) State the Schoenberg-Whitney Theorem.

(b) Choose a knot sequence $\mathbf{t}$ so that Λ'_τ maps $S_{3,\mathbf{t}}$ 1-1 onto $\mathbb{R}^n$ (explain).

(c) What practical purpose could the resulting spline space $S_{3,\mathbf{t}}$ serve?

(5) Let $X = \mathbb{R}^I$ be the linear space of all real-valued functions on the interval $I := [0\,.\,.\,4]$, and consider the linear map $V : \mathbb{R}^3 \to X : a \mapsto \sum_{i=1}^3 v_i a(i)$ with

$$v_1(s) := (s-2)(s-3), \quad v_2(s) := (s-3)(s-1), \quad v_3(s) := (s-1)(s-2).$$

(a) Prove that V is 1-1.

(b) Prove that $\operatorname{ran} V = \Pi_2$.

(c) Construct a linear map $\Lambda' : X \to \mathbb{R}^3$ for which $\Lambda' V = \mathrm{id}$.

(note:) You may prefer to reorder this list of requests.

(d) Which of these maps is an *analysis operator*, a *synthesis operator*, a *reconstruction map*, a *data map*? Explain.