CS717 Spring 06
Prof. Amos Ron

Answers to homework problems
Provided by Prof. Carl de Boor

I.7 While it is necessary, it is not sufficient to verify that $\operatorname{ran} A$ is closed under vector addition and scalar multiplication since $\operatorname{ran} A$ is not, offhand, a subset of a ls. On the other hand, there is no call for *defining* the vector operations on $\operatorname{ran} A$ since they are already defined on all of U. Verification is needed and is straightforward. E.g., $A0$ can serve as the neutral element since, for all $Ax \in \operatorname{ran} A$, $Ax + A0 = A(x+0) = Ax$, while, for every $Ax \in \operatorname{ran} A$, $A(-x) = -Ax$ can serve as the inverse since $Ax + A(-x) = A(x-x) = A0$. Etc.

I.11 (a) the cylinder spanned by two disks, radius 1, center 0 and center (1,1) ; (b) the octahedron with vertices (0,0), (1,0), (2,1), (2,2), (1,3), (0,3), (-1,2), (-1,1); (c) union of $[0\,..\,1]^2$ shifted by (-1,-1) with that shifted by (-2,-2).

I.16 The main work here is to prove that X is finitely generated, i.e., has a basis, hence has dimension. The slick way to prove this is to observe that the collection of all 1-1 column maps into X is not empty: it contains the unique linear map from $\mathbb{F}^0$ to X. By (12)Lemma, $\dim Y$ is a (finite) upper bound on the number of columns in any 1-1 column map into $X \subseteq Y$, therefore, there is a 1-1 column into X with a maximal number of columns. Any such V must be a basis for X, by remark following (8)Lemma. Thus $\dim X = \#V \le \dim Y$. If now $X \ne Y$, then there exists $y \in Y \backslash X$, hence, by (8)Lemma, $[V, y]$ is 1-1 into Y, hence, by (12)Lemma, $\dim X = \#V < \#V + 1 \le \dim Y$.

I.20 With $X := \Pi_2(\mathbb{R}^2)$, the set $S := \{p \in X : p|_T = 0\}$ is the kernel of the lm $A : X \to \mathbb{R}^T : p \mapsto p|_T$.

Since T lies on some straight line, there is some nontrivial vector n normal to that straight line, and then, with $a \in T$, $\ell : x \mapsto n^t(x-a)$ is a linear polynomial that vanishes on that line (and nowhere else). It follows that the map $f \mapsto f\ell$ (:= pointwise product $f\ell : x \mapsto f(x)\ell(x)$) carries $\Pi_1(\mathbb{R}^2)$ into S. Since it is also linear and 1-1, we get $\dim S \ge \dim \Pi_1(\mathbb{R}^2) = 3$.

On the other hand, with m some 2-vector not parallel to n, $\ell_t : x \mapsto m^t(x-a)$ is a linear polynomial that, assuming $a \in T$, vanishes at a but at no other point in T. Since the (pointwise) product of any two of these is in X and $\#T = 4$, it follows that $\operatorname{ran} A$ contains the vectors $(\times, 0, 0, \times)$, $(0, \times, 0, \times)$, and $(0, 0, \times, \times)$ where $\times$ stands for something nonzero, hence $\dim \operatorname{ran} A \ge 3$, while $\dim \operatorname{dom} A = \dim \Pi_2(\mathbb{R}^2) = 6$.

So, with the Dimension Formula,

$$3 \le \dim S = \dim \ker A = \dim \operatorname{dom} A - \dim \operatorname{ran} A \le 3,$$

showing $\dim S = 3$ (and also that $\dim \operatorname{ran} A = 3$).

I.21 $\Lambda^t V = 1$ implies that V is 1-1, i.e., its columns are linearly independent. If $\sum_j c_i \lambda_i = 0$ (with $\lambda_i := \delta_0 D^{i-1}$), then $c^t = c^t 1 = c^t(\Lambda^t V) = (c^t \Lambda^t)V = 0V = 0$, hence $\lambda_1, \ldots, \lambda_m$ must be linearly independent, too.

I.23 You are taught to differentiate polynomials in exactly one way: write the polynomial as a linear combination of powers, then apply the rule $D()^j = j()^{j-1}$ to the individual powers, multiply by the power coefficients, then sum.

This is exactly the statement that $D|_{\Pi_k} = VAV^{-1}$, with $V = [()^0, ()^1, \ldots, ()^k] : \mathbb{F}^{k+1} \to \Pi_k$, and $A = [0, e_1, 2e_2, \ldots, ke_k] \in \mathbb{F}^{(k+1)\times(k+1)}$.

I.28 This is a throw-away since it only tests whether you actually read the notes. Since $n = \dim \operatorname{ran} A = \operatorname{rank} A$, we know that $A = V\Lambda^t$ with $V \in L(\mathbb{F}^n, U)$ 1-1. Since $A' = \Lambda V'$ with $\Lambda \in L(\mathbb{F}^n, X')$ and $\operatorname{rank} A' = \operatorname{rank} A = n$, it follows that $\Lambda V'$ is a minimal factorization for A', hence also Λ must be 1-1.

Of course, starting from scratch, since $\dim \operatorname{ran} A = n$, there is a basis $V \in L(\mathbb{F}^n, \operatorname{ran} A)$, hence $A = V\Lambda^t$ with $\Lambda^t := V^{-1}A$ onto since, necessarily, $V = AW$ for some W, and so $\Lambda^t W = V^{-1}AW = 1$.

I.29 For $\lambda_i = \delta_{1/i}$, $i \in \mathbb{N}$, $\ker \delta_0 \supseteq \cap_i \ker \lambda_i$ (since $f(0) = \lim_{i\to\infty} f(1/i)$ for every $f \in C([0\,..\,1])$). Yet, for any n, $\prod_{i<n}(\cdot - 1/i) \in \cap_{i<n} \ker \lambda_i \backslash \ker \delta_0$, hence $\delta_0 \notin \operatorname{ran}[\lambda_i : i \in \mathbb{N}]$.

I.30 '$\Rightarrow$': By (23)Proposition, V is 1-1, as is Λ since then also $A' = \Lambda V'$ is minimal, therefore Λ^t is onto.

'$\Leftarrow$': Let $n := \#V$. Since Λ^t is onto, $\operatorname{ran} A = V(\mathbb{F}^n) = \operatorname{ran} V$, and, since V is 1-1, V is a basis for $\operatorname{ran} V = \operatorname{ran} A$, hence $A = V\Lambda^t$ is minimal, by (23)Proposition.

I.31 Let Λ be a basis for L,hence $L\perp = \ker \Lambda^t$ and, by (31)Lemma, $\operatorname{ran} \Lambda = \perp \ker \Lambda^t$, i.e., $\perp(L\perp) = L$. H.P.(29) gives example with $\perp(L\perp)$ much greater than L.