I.7 While it is necessary, it is not sufficient to verify that \( \text{ran}\ A \) is closed under vector addition and scalar multiplication since \( \text{ran}\ A \) is not, offhand, a subset of a \( \Lambda \). On the other hand, there is no call for defining the vector operations on \( \text{ran}\ A \) since they are already defined on all of \( U \). Verification is needed and is straightforward. E.g., \( A0 \) can serve as the neutral element since, for all \( Ax \in \text{ran}\ A \), \( Ax + A0 = A(x + 0) = Ax \), while, for every \( Ax \in \text{ran}\ A \), \( A(-x) = -Ax \) can serve as the inverse since \( Ax + A(-x) = A(x - x) = A0 \). Etc.

I.11 (a) the cylinder spanned by two disks, radius 1, center 0 and center \((1,1)\); (b) the octahedron with vertices \((0,0)\), \((1,0)\), \((2,1)\), \((2,2)\), \((1,3)\), \((0,3)\), \((-1,2)\), \((-1,1)\); (c) union of \([0\ldots1]^2\) shifted by \((-1,-1)\) with that shifted by \((-2,-2)\).

I.16 The main work here is to prove that \( X \) is finitely generated, i.e., has a basis, hence has dimension. The slick way to prove this is to observe that the collection of all 1-1 column maps into \( X \) is not empty: it contains the unique linear map from \( \Pi^0 \) to \( X \). By (12)Lemma, \( \dim Y \) is a (finite) upper bound on the number of columns in any 1-1 column map into \( X \subseteq Y \), therefore, there is a 1-1 column into \( X \) with a maximal number of columns. Any such \( V \) must be a basis for \( X \), by remark following (8)Lemma. Thus \( \dim X = \#V \leq \dim Y \). If now \( X \neq Y \), then there exists \( y \in Y \setminus X \), hence, by (8)Lemma, \( [V,y] \) is 1-1 into \( Y \), hence, by (12)Lemma, \( \dim X = \#V < \#V + 1 \leq \dim Y \).

I.20 With \( X := \Pi_2(\mathbb{R}^2) \), the set \( S := \{ p \in X : p|_T = 0 \} \) is the kernel of the lm \( A : X \to \mathbb{R}^T : p \mapsto p|_T \).

Since \( T \) lies on some straight line, there is some nontrivial vector \( n \) normal to that straight line, and then, with \( a \in T \), \( \ell : x \mapsto n^t(x - a) \) is a linear polynomial that vanishes on that line (and nowhere else). It follows that the map \( f \mapsto f\ell \) (\( := \) pointwise product \( f\ell : x \mapsto f(x)\ell(x) \)) carries \( \Pi_1(\mathbb{R}^2) \) into \( S \). Since it is also linear and 1-1, we get \( \dim S \geq \dim \Pi_1(\mathbb{R}^2) = 3 \).

On the other hand, with \( m \) some 2-vector not parallel to \( n \), \( \ell_m : x \mapsto m^t(x - a) \) is a linear polynomial that, assuming \( a \in T \), vanishes at \( a \) but at no other point in \( T \). Since the (pointwise) product of any two of these is in \( X \) and \( \#T = 4 \), it follows that \( \text{ran}\ A \) contains the vectors \((x,0,0,\times),(0,x,0,\times),\) and \((0,0,\times,\times)\) where \( \times \) stands for something nonzero, hence \( \dim \text{ran}\ A \geq 3 \), while \( \dim \text{dom}\ A = \dim \Pi_2(\mathbb{R}^2) = 6 \).

So, with the Dimension Formula,

\[ 3 \leq \dim S = \dim \ker A = \dim \text{dom}\ A - \dim \text{ran}\ A \leq 3, \]

showing \( \dim S = 3 \) (and also that \( \dim \text{ran}\ A = 3 \)).

I.21 \( \Lambda^tV = 1 \) implies that \( V \) is 1-1, i.e., its columns are linearly independent. If \( \sum_j c_i\lambda_i = 0 \) (with \( \lambda : = \delta_0D^{i-1} \)), then \( c^t1 = c^t(\Lambda^tV) = (c^t\Lambda^t)V = 0V = 0 \), hence \( \lambda_1,\ldots,\lambda_m \) must be linearly independent, too.
I.23 You are taught to differentiate polynomials in exactly one way: write the polynomial as a linear combination of powers, then apply the rule $D(j) = j(j-1)$ to the individual powers, multiply by the power coefficients, then sum.

This is exactly the statement that $D_{\Pi_k} = VAV^{-1}$, with $V = [(0, 1, \ldots, k)] : \mathbb{F}^{k+1} \to \Pi_k$, and $A = [0, e1, 2e2, \ldots, kek] \in \mathbb{F}^{(k+1)\times(k+1)}$.

I.28 This is a throw-away since it only tests whether you actually read the notes. Since $n = \dim \text{ran} A = \text{rank} A$, we know that $A = VA^t$ with $V \in L(\mathbb{F}^n, U)$ 1-1. Since $A' = AV'$ with $A \in L(\mathbb{F}^n, X)$ and $\text{rank} A' = \text{rank} A = n$, it follows that $AV'$ is a minimal factorization for $A'$, hence also $A$ must be 1-1.

Of course, starting from scratch, since $\dim \text{ran} A = n$, there is a basis $V \in L(\mathbb{F}^n, \text{ran} A)$ hence $A = VA^t$ with $A^t := V^{-1}A$ onto since, necessarily, $V = AW$ for some $W$, and so $A^tW = V^{-1}AW = 1$.

I.29 For $\lambda_i = \delta_{1,i}$, $i \in \mathbb{N}$, $\ker \delta_0 \supseteq \cap_i \ker \lambda_i$ (since $f(0) = \lim_{i \to \infty} f(1/i)$ for every $f \in C([0, 1])$). Yet, for any $n$, $\prod_{i < n} (-1/i) \in \cap_i \ker \lambda_i \setminus \ker \delta_0$, hence $\delta_0 \notin \text{ran}[\lambda_i : i \in \mathbb{N}]$.

I.30 $'\Rightarrow'$: By (23)Proposition, $V$ is 1-1, as is $A$ since then also $A' = AV'$ is minimal, therefore $A^t$ is onto.

$'\Leftarrow'$: Let $n := \#V$. Since $A^t$ is onto, $\text{ran} A = V(\mathbb{F}^n) = \text{ran} V$, and, since $V$ is 1-1, $V$ is a basis for $V = \text{ran} A$, hence $A = VA^t$ is minimal, by (23)Proposition.

I.31 Let $\Lambda$ be a basis for $L$, hence $L^\perp = \ker \Lambda^t$ and, by (31)Lemma, $\text{ran} \Lambda = \perp \ker \Lambda^t$, i.e., $\perp(L^\perp) = L$. H.P.(29) gives example with $\perp(L^\perp)$ much greater than $L$. 

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