CS717 Spring 06
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Comments to Assignments #3 and #4
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I.37 Let $t_j := a + jh$, all j, and $h := (b-a)/N$. Let v_j be the function with support in $[t_{j-1} \mathinner{\ldotp\ldotp} t_{j+1}]$, and $v_j = (\cdot - t_{j-1})/\nabla t_j$ on $[t_{j-1} \mathinner{\ldotp\ldotp} t_j]$ and $v_j = (t_{j+1} - \cdot)/\Delta t_j$ on $[t_j \mathinner{\ldotp\ldotp} t_{j+1}]$. Let $\Lambda^t : g \mapsto (g(t_j) : j = 0, \ldots, N)$. Then $\Lambda^t V = 1$ and $\mu V = (1/2, 1, \ldots, 1, 1/2)h$, hence $\mu V \Lambda^t = c^t \Lambda^t$.

If you are willing to consider also piecewise continuous functions, then another choice would be $v_j = \chi_{(t_{j-1/2} \mathinner{\ldotp\ldotp} t_{j+1/2})}$, all j, with $t_{j+1/2} := \max(a, \min(b, (t_j + t_{j+1})/2))$, all j.

II.1 Since $\mathbf{B}(t)$ refines both $\mathbf{A}$ and $\mathbf{B}(t)$, it refines their union, $\mathbf{B}'(t)$, which contains $\mathbf{B}(t)$, hence refines it.

II.3 (a). For each $B_{r,R}(f)$ in $\mathbf{B}_{pw}(f)$, trivially $B_r(f) \subset B_{r,R}(f)$ since $R \subset S$. For the other direction, let $B_r(f)$ be any element in $\mathbf{B}_\infty(f)$. There is no $B_{r',R}(f)$ with $r' > 0$, $R \subset S$ and $\#R < \infty$, since we can always find a g such that $g \in B_{r',R}(f)$ but $g \notin B_r(f)$ (e.g., let $g(s) := f(s)$ if $s \in R$, and $g(s) := f(s) + r$ if $s \in S \setminus R$).

(b). Note that (f_n) converges to f uniformly in $\mathbb{R}^S$ if for any $\epsilon > 0$, there exists N s.t. $\sup_{s \in S} |f_n(s) - f(s)| < \epsilon$, $\forall n > N$. This condition is equivalent to $f(> N) \subset B_\epsilon(f)$.

Note that (f_n) converges to f pointwise in $\mathbb{R}^S$ if for every $s \in S$ and for any $\epsilon > 0$, there exists $N(s)$ s.t. $|f_n(s) - f(s)| < \epsilon$, $\forall n > N(s)$. This condition is equivalent to $f(> N(s)) \subset B_{\epsilon,\{s\}}(f)$. Finally due to **(2) Neighborhood Assumption 1**, this last condition is equivalent to $f(> \max_{s \in R} N(s)) \subset B_{\epsilon,R}(f)$ for any $R \subset S$ with $\#R < \infty$.

II.4 Each $\mathbf{B}(t)$ consists of exactly one set, hence (1) is trivially satisfied, and this set contains t. Take the set $U := \{1, 2\}$. Then $U^o = \{1\}$ (since the sole neighborhood of 1 is U, hence lies entirely in U, while the sole neighborhood of 2 does not lie in U). But the interior of $U^o = \{1\}$ is empty (since the sole neighborhood of its sole element, 1, fails to lie in U^o), therefore $(U^o)^o \neq U^o$, i.e., the interior of U fails to be open.

This does not contradict H.P.(II.8) since our proposed nbhdsystem fails to satisfy (5): For, our U is the only nbhd the point 1 has, but U fails to be open (since it does not equal its interior), nor is its sole proper subset containing 1, namely U^o, open, as we just saw. In other words, U is a neighborhood of 1 that fails to contain an open set containing 1.

II.9 $\|x - y\| \leq \|x - z\| + \|z - y\|$ implies that $d(x, Y) \leq \|x - z\| + d(z, Y)$, hence, by symmetry, $|d(x, Y) - d(z, Y)| \leq \|x - z\|$, i.e., $x \mapsto d(x, Y)$ is even Lipschitz-continuous with constant 1. This makes $B_r(Y)$ ($B_r^-(Y)$) the inverse image of an open (a closed) set under a continuous map, hence open (closed).

If X is the integers with the absolute value metric, then $B_1(0) = \{0\} = B_1(0)^-$, while $B_1^-(0) = \{-1, 0, 1\}$.

II.15 The interpolant to f at points a and b has the form $P_{a,b}f := f(a)l_{a,b} + f(b)l_{b,a}$, with $l_{a,b} : t \mapsto (x - b)/(a - b)$. Hence $f(x) - P_{a,b}f(x) = (f(x) - f(a))l_{a,b}(x) + (f(x) - f(b))l_{b,a}(x)$, therefore, for $a \leq x \leq b$, $|f(x) - P_{a,b}f(x)| \leq |f(a) - f(x)|l_{a,b}(x) + |f(x) - f(b)|l_{b,a}(x) \leq \omega_f(|b - a|)l_{a,b} + \omega_f(|b - a|)l_{b,a}(x) = \omega_f(|b - a|)$, using (twice) the fact that $l_{a,b} + l_{b,a} = ()^0$ and (once) the fact that both $l_{a,b}(x)$ and $l_{b,a}(x)$ are nonnegative for $a \leq x \leq b$.

Since, on each interval $[t_j \mathinner{..} t_{j+1}]$, $P_t f$ agrees with $P_{t_j,t_{j+1}} f$ and ω_f is nondecreasing, it follows that $|f(x) - P_t f(x)| \le \omega_f(\max_j |t_j - t_{j+1}|)$.

II.19 If f is the pointwise limit of some sequence (f_n) in $X_c \supset X_0$, then $\operatorname{supp} f \subset \cup_n \operatorname{supp} f_n$, hence is countable (as the countable union of countable sets). Conclusion: X_c is sequentially closed. Also, for any $f \in X_c \backslash X_0$, there exists an enumeration $(t_n : n \in \mathbb{N})$ of its support, and f is the pointwise limit of the sequence $(f\chi_{\{t_1,\ldots,t_n\}} : n \in \mathbb{N})$ in X_0. Conclusion, X_c is the sequential closure of X_0. Finally, since T is not countable, X_c is a proper subset of X.

On the other hand, for every $f \in X$, every $r > 0$, and every finite $S \subset T$, the function g on T that agrees with f on S and is zero otherwise, is both in X_0 and in the pw ball $B_{r,S}(f)$ (showing that every pw nbhd of f meets X_0). Hence, X_0 is dense in X.

II.22 '$\Rightarrow$': Since the A_n are not empty, there exists $a_n \in A_n$, all n, and, since (A_n) is decreasing, $a_{>n} \subseteq A_n$, hence $\operatorname{diam}(a_{>n})^- \le \operatorname{diam} A_n \to 0$, hence (a_n) is Cauchy, hence has a limit, a_∞ say, and, since A_n is closed, $a_\infty \in A_n$, all n.

'$\Leftarrow$': Let (a_n) be a Cauchy sequence. Then $A_n := (a_{>n})^-$ is a decreasing sequence of closed sets with $\operatorname{diam} A_n \to 0$, hence there exists $a_\infty \in \cap_n (a_{>n})^-$, and $d(a_n, a_\infty) \le \operatorname{diam} A_n \to 0$ as $n \to \infty$, i.e., $a_\infty = \lim_n a_n$.

II.24 Any limit point of a sequence has points from that sequence in any of its neighborhoods, hence is in the closure of the range of the sequence, but so is every point in the sequence, even those that are isolated, as is, e.g., the case for all the entries of the sequence $n \mapsto 1/n$ in $\mathbb{R}$.

II.25 (i) If z is limit point of (x_n), then $d(z, x_{n+1}) \le d(z, x_{>n}) + \operatorname{diam}(x_{>n}) = \operatorname{diam}(x_{>n})$, hence, if also (x_n) is Cauchy, then $d(z, x_{n+1}) \to 0$, i.e., $\lim x_n = z$.

(ii) If (y_n) is a subsequence of (x_n), i.e., $y_n = x_{\mu(n)}$ for some strictly increasing μ, then $\{y_{>n}\} \succ \{x_{>n}\}$, hence any limit point of (y_n) is a limit point of (x_n). On the other hand, as observed above, if (x_n) has a limit point, then some subsequence converges to it. So, if all subsequences have the same nonempty set of limit points, then they all must have the same point as their sole limit point, z. Yet, if (x_n) were not to converge to it, i.e., for some $r > 0$, every $x_{>n}$ were to fail to get inside $B_r(z)$, then we could find inductively a strictly increasing $\mu : \mathbb{N} \to \mathbb{N}$ with $d(z, x_{\mu(n)}) \ge \varepsilon$, hence z could not be a limit point for this subsequence, a contradiction. On the other hand, if (x_n) converges, then all its subsequences converge to the same limit, z, hence they all have the same nonempty set of limit points, $\{z\}$.