Problem 1. Let $\Phi := (\phi_j)_{j=1}^{\infty}$ be a subset of a Hilbert space $X$, and let $\Psi = (\psi_j)_{j=1}^{\infty}$ be a complete orthonormal basis for $X$ (an HS which has a complete orthonormal basis is called separable.) Consider the following two properties of $\Phi$:

Property 1. There exists a map $A \in bL(X)$ which maps $X$ 1-1 onto itself, and maps $\Psi$ 1-1 onto $\Phi$: $\phi_j = A\psi_j, j = 1, 2, \ldots$.

Property 2. The map $T^* := T_\Phi^* : X \to \ell_2(\Phi) : x \mapsto (\langle x, \phi \rangle)_{\phi \in \Phi}$

is well-defined, 1-1 and onto (and hence invertible by the OMT). In particular, there exist two positive constants $C_1, C_2$ such that

$$C_1\|x\| \leq \|T^*x\|_{\ell_2} \leq C_2\|x\|, \quad x \in X.$$  

(a) Prove that Property 1 implies Property 2. (The two properties are actually equivalent. Each defines the notion of a Riesz basis).

Proof: Property 1 implies Property 2: $A$ is bounded 1-1. Since it is onto, it has a closed, hence complete, range. Therefore, it is boundedly invertible, by virtue of the OMT. This implies that the dual map $A^*$ is also bounded, 1-1, onto and boundedly invertible. Now, $\langle x, A\psi \rangle = \langle A^*x, \psi \rangle$, and hence $T_\Phi^* = T_\Psi^* A^*$. Since $\Psi$ is o.n., $T_\Psi^*$ is bounded, boundedly invertible, 1-1 and onto. Consequently, $T_\Phi^*$ has all these requisite properties, too.

(b) Prove also that, given a Riesz basis $\Phi$ (defined by Property 1), there exists another Riesz basis $\tilde{\Phi} = (\tilde{\phi}_j)_{j=1}^{\infty}$ such that, for every $x \in X$, the series

$$\sum_{j=1}^{\infty} \langle x, \phi_j \rangle \tilde{\phi}_j$$

converges to $x$.

Proof: Define: $\tilde{\phi}_j := (A^*)^{-1}\psi_j, \psi_j \in \Psi$. Now, for every $x \in X$,

$$x = (A^*)^{-1}(A^*x) = A^{*-1}(\sum_{\psi \in \Psi} \langle A^*x, \psi \rangle \psi) = (\sum_{\psi \in \Psi} \langle x, A\psi \rangle A^{*-1}\psi).$$

Here, we expanded $A^*x$ in the o.n. $\Psi$ (2nd equality), and used the convergence of the summation in $X$ and the continuity of $A^{*-1}$ (3rd equality).

$\tilde{\Phi}$ is Riesz, since $A^{*-1}$ has all the requisite properties that are stipulated in Property 1. 

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Then prove that \( A \) is 1-1 then the OMT guarantees the boundedness of \( A \) is also quite simple: If \( AX = 0 \), then \( X = 0 \).

Problem 2. Let \( X, Y \) be two Hilbert spaces. Recall that a map \( A \in L(X, Y) \) is called unitary if \( \|Ax\| = \|x\| \), for every \( x \in X \).

(a) Give a non-constructive proof that a unitary map is left invertible, i.e., that there exists \( C \in bL(Y, X) \) such that \( CA = 1 \). Do this part by showing first that \( A \) is 1-1 and has closed range.

Proof: If \( A \) has indeed, closed range \( Z \), then the range \( Z \) is Hilbert, too. If \( A \) is also 1-1 then the OMT guarantees the boundedness of \( A^{-1} \in L(Z, X) \). Thus, we only need to show that \( A \) is 1-1 and has a closed range. The fact that \( A \) is 1-1 is trivial. The closed range is also quite simple: If \( (Ax_n)_n \) converges in \( Y \), then it is Cauchy. Since \( A \) is unitary \( (x_n)_n \) is Cauchy. Since \( X \) is Hilbert, \( (x_n)_n \) converges, say to \( x \). Since \( A \) is continuous, \( (Ax_n)_n \) converges to \( Ax \), hence ran \( A \) is closed.

(b) Give a constructive proof that a unitary map is invertible by showing that \( A^* A = 1 \). Hint: use the definition of \( A^* \) in ips: \( A^* y \) in the linear functional in \( X^* = X \) that satisfies
\[
\langle x, A^* y \rangle := \langle Ax, y \rangle, \quad x \in X.
\]
Then prove that \( \langle (1 - A^* A)x, x \rangle = 0 \), for every \( x \in X \). Then try to conclude that \( \langle (1 - A^* A)x, x' \rangle = 0 \), for every \( x, x' \in X \).

Proof: \( \langle (1 - A^* A)x, x \rangle = \langle x, x \rangle - \langle Ax, Ax \rangle = 0 \), since \( A \) is unitary. Now, let \( x, y \) be arbitrary, \( C := 1 - A^* A \). First, we have \( C^* = C \), since \( \langle A^* Ax, y \rangle = \langle Ax, Ay \rangle = \langle x, A^* Ay \rangle = \langle (A^* A)^* x, y \rangle \). Then
\[
0 = \langle C(x + y), x + y \rangle = \langle Cx, x \rangle + \langle Cy, y \rangle + \langle Cx, y \rangle + \langle Cy, x \rangle.
\]
The first two summands equal 0, and since \( \langle Cy, x \rangle = \langle y, C^* x \rangle = \langle y, Cx \rangle \), we conclude that \( \text{Re}(Cx, y) = 0 \). Choosing \( y := Cx \), shows that \( Cx = 0 \), hence \( C = 0 \), hence \( 1 = A^* A \), as claimed.

\( \square \)

Problem 3. Let \( A \) be a unitary map as in Problem 2.

(a) Show that \( AA^* \) is the orthogonal projector of \( Y \) onto ran \( A \). From that, derive a necessary and sufficient condition for \( AA^* \) to the identity.


Thus, \( C := AA^* \) is a projector. As in Problem 2, \( C^* = C \). To prove that \( C \) is orthogonal, we need to prove that \( \ker C \perp \text{ran} \ C \). This is true since, if \( y \in \ker C \), then
\[
\langleCx, y\rangle = \langle x, Cy\rangle = 0.
\]
To prove that \( \text{ran} \ C = \text{ran} A \) it is sufficient to show that \( \ker C = \ker A^* \). Obviously, \( \ker C \supset \ker A^* \). Now, if \( y \in \ker C \), then
\[
0 = \langle y, Cy\rangle = \langle A^* y, A^* y\rangle,
\]
hence \( A^* y = 0 \).

Now, \( AA^* = 1 \) iff the projector in onto the entire space. Since the range of the projector is \( \text{ran} A \) the condition we need is \( \text{ran} A = Y \).

\( \square \)
(b) Recall that a set \( \Phi = (\phi_j)_{j=1}^{\infty} \) of an \( H_s \) is a tight frame if the map
\[
T^* : X \to \ell_2(\Phi) : x \mapsto (\langle x, \phi \rangle)_{\phi \in \Phi}
\]
is unitary. Show that for \( x \in X \) the sequence \( T^*x \) has minimal norm among all sequences \( a \in \ell_2(\Phi) \) that satisfy
\[
x = \sum_{\phi \in \Phi} a(\phi) \phi.
\]
(1)

**Hint:** Take \( A := T^* \) in (a), and compute explicitly \( A^* \). Then use the result in (a). Your proof should show that the sum in (1) converges in \( X \) regardless of the order of the elements in \( \Phi \).

**Proof:** With \( A := T^* \), we know that \( A \) is unitary. Fix \( f \in \Phi \) and define \( \delta \in \ell_2(\Phi) \) to be the sequence that assumes the value 1 at \( f \) and 0 elsewhere. Then, for \( x \in X \),
\[
\langle A^*\delta, x \rangle = \langle \delta, Ax \rangle = \langle f, x \rangle.
\]
This shows that \( A^*\delta = f \), hence, by linearity, that
\[
A^*a = \sum_{\phi \in \Phi} a(\phi) \phi.
\]
Now, if \( x = A^*a \) for some \( x \in X \) and \( a \in \ell_2(\Phi) \), then, by (a), \( Ax = (AA^*)a \) is the orthogonal projection of \( a \) on \( \text{ran} \ A \), hence \( \|Ax\| \leq \|a\| \).

**Problem 4.** Recall that a compactly supported \( w \in L_2(\mathbb{R}) \) has \( m \) vanishing moments if
\[
\int_{\mathbb{R}} w(t)p(t) \, dt = 0,
\]
for every polynomial \( p \in \Pi_{m-1} \). Recall that we proved in class that if \( \text{supp} \ w \subset [A,B] \), and if \( f \in C^{(m)}([A,B]) \), then
\[
|\langle f, w \rangle| \leq \frac{1}{m!} \|D^m f\|_{L_\infty([A,B])} \|w\| (B - A)^{m+1/2}.
\]

Now, assume that \( f \in C^{(m)}(\mathbb{R}) \), and, in addition, \( \text{supp} \ f \subset [0,1] \). Assume also that \( \text{supp} \ w = [-A,A] \) for some integer \( A > 0 \), that \( \|w\| = 1 \), and that \( w \) is bounded (and that, as before, \( w \) has \( m \) vanishing moments). Recall that, for \( j,k \in \mathbb{Z} \), \( w_{j,k} \) is defined by
\[
w_{j,k} : t \mapsto 2^{j/2} w(2^j t - k).
\]
For every fixed \( j \in \mathbb{Z} \), provide an estimate on the \( \ell_2(\mathbb{Z}) \)-norm of the sequence
\[
k \mapsto \langle f, w_{j,k} \rangle.
\]
I.e., estimate \( (\sum_{k \in \mathbb{Z}} |\langle f, w_{j,k} \rangle|^2)^{1/2} \). Hint: consider separately the case of \( j < 0 \) and the case \( j \geq 0 \).

**Proof:** For \( j \geq 0 \), the support of \( w_{j,k} \) intersect \([0, 1]\) only if \(-A < k < 2^j + A \). Also, \( \|w_{j,k}\| = \|w\| = 1 \). The estimate above implies that, with \( C := \frac{\|D^m f\|_{L_\infty(\mathbb{R})}}{m!} (2A)^{2m+1} \),

\[
A_j^2 := \sum_{k=-\infty}^{\infty} |\langle f, w_{j,k} \rangle|^2 \leq \sum_{k=-A+1}^{2^j + A - 1} \frac{\|D^m f\|_{L_\infty(\mathbb{R})}}{m!} \left( \frac{2A}{2^j} \right)^{2m+1} = C (2^j + 2A - 1) 2^{-(2m+1)j}.
\]

For \( j \geq \log_2(A) \), we can estimate

\[ A_j \leq 2\sqrt{C} 2^{-mj}. \]

So, while the number of non-zero entries grows exponentially with \( j \), the total “energy” \( A_j \) decays exponentially with \( j! \).

For \( j < 0 \), there are \( 2A \) values of \( k \) for which \( \text{supp} w_{j,k} \) overlaps with \([0, 1]\). We can estimate \( \langle w_{j,k}, f \rangle \) for each such \( k \) by \( \|w_{j,k}\|_{L_\infty} \|f\|_{L_1([0,1])} \). Setting

\[ K := \|w\|_{L_\infty} \|f\|_{L_1([0,1])}, \]

we know that \( K \) is finite, since \( w \) is bounded. We also know that \( \|w_{j,k}\|_{L_\infty} = 2^{j/2} \|w\| \).

The final bound in this case is then

\[ A_j \leq \sqrt{2AK} 2^{j/2}. \]

This also goes to zero exponentially fast (as \( j \to -\infty \)), but not as nearly as fast as the other case.

**Bonus:** Let’s see that the wavelet representation of \( f \) is sparse. For \( n > 0 \), let’s try to estimate the number \( N(n) \) of wavelet coefficients \( \langle f, w_{j,k} \rangle \) whose modulus is \( \geq 2^{-n} \). We expect this number to grow exponentially with \( n \), i.e., to be behave like \( \text{const} 2^{n\alpha} \). We are interested in finding the parameter \( \alpha \), which can be fetched by computing \( \limsup_{n} \frac{\log_2(N(n))}{n} \).

The \( \limsup \) above does not depend on \( A, C \) and \( K \) so, for convenience, we chose \( K = C = 1 \) and \( A = 1/2 \).

For \( j \geq 0 \), \( |\langle w_{j,k}, f \rangle| \leq 2^{-(m+.5)j} \). Hence, if \( j > n/(m+.5) \), all the wavelet coefficients are too small. The total number of wavelet coefficients \( \langle w_{j,k}, f \rangle \), \( j \geq 0 \), that need to be counted is about \( 2^{n/(m+.5)} \) (why?). For \( j < 0 \), the wavelet coefficients fall below the threshold once \( j < -2n \). The total number here is about \( 2n \) (should be \( 2A \times 2n \), but we assume \( 2A = 1 \)). This number is negligible compared to the first one. So, \( \alpha = 1/(m+.5) \) here. Everything depends on the vanishing moments (and the smoothness of \( f \))!\[ \Box \]