CS717 Spring 06
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Comments on Assignment #11

Problem 1. *Let* $\Phi := (\phi_j)_{j=1}^\infty$ *be a subset of a Hilbert space* X*, and let* $\Psi = (\psi_j)_{j=1}^\infty$ *be a complete orthonormal basis for* X *(an Hs which has a complete orthonormal basis is called separable.) Consider the following two properties of* Φ*:*

Property 1. There exists a map $A \in bL(X)$ *which maps* X *1-1 onto itself, and maps* Ψ *1-1 onto* Φ*:* $\phi_j = A\psi_j$, $j = 1, 2, \ldots$.

Property 2. The map

$$T^* := T^*_\Phi : X \to \ell_2(\Phi) : x \mapsto (\langle x, \phi\rangle)_{\phi\in\Phi}$$

is well-defined, 1-1 and onto (and hence invertible by the OMT). In particular, there exist two positive constants C_1*,* C_2 *such that*

$$C_1\|x\| \le \|T^* x\|_{\ell_2} \le C_2\|x\|, \quad x \in X.$$

(a) Prove that Property 1 implies Property 2. (The two properties are actually equivalent. Each defines the notion of a Riesz basis).

Proof: **Property 1 implies Property 2:** A is bounded 1-1. Since it is onto, it has a closed, hence complete, range. Therefore, it is boundedly invertible, by virtue of the OMT. This implies that the dual map A^* is also bounded, 1-1, onto and boundedly invertible. Now, $\langle x, A\psi\rangle = \langle A^* x, \psi\rangle$, and hence $T^*_\Phi = T^*_\Psi A^*$. Since Ψ is o.n., T^*_Ψ is bounded, boundedly invertible, 1-1 and onto. Consequently, T^*_Φ has all these requisite properties, too. □

(b) Prove also that, given a Riesz basis Φ *(defined by Property 1), there exists another Riesz basis* $\widetilde\Phi = (\widetilde\phi_j)_{j=1}^\infty$ *such that, for every* $x \in X$*, the series*

$$\sum_{j=1}^\infty \langle x, \phi_j\rangle\, \widetilde\phi_j$$

converges to x*.*

Proof: Define: $\widetilde\phi_j := (A^*)^{-1}\psi_j$, $\psi_j \in \Psi$. Now, for every $x \in X$,

$$x = (A^*)^{-1}(A^* x) = A^{*-1}\Big(\sum_{\psi\in\Psi}\langle A^* x, \psi\rangle\psi\Big) = \Big(\sum_{\psi\in\Psi}\langle x, A\psi\rangle A^{*-1}\psi\Big).$$

Here, we expanded $A^* x$ in the o.n. Ψ (2nd equality), and used the convergence of the summation in X and the continuity of A^{*-1} (3rd equality).

$\widetilde\Phi$ is Riesz, since A^{*-1} has all the requisite properties that are stipulated in Property 1. □

Note: the order of the basis elements in a Riesz basis is immaterial. The convergence is valid regardless of any ordering. This is in stark contrast with the weaker notion of a Schauder basis.

Problem 2. *Let X, Y be two Hilbert spaces. Recall that a map $A \in L(X, Y)$ is called unitary if $\|Ax\| = \|x\|$, for every $x \in X$.*

(a) Give a non-constructive proof that a unitary map is left invertible, i.e., that there exists $C \in bL(Y, X)$ such that $CA = 1$. Do this part by showing first that A is 1-1 and has closed range.

Proof: If A has indeed, closed range Z, then the range Z is Hilbert, too. If A is also 1-1 then the OMT guarantees the boundedness of $A^{-1} \in L(Z, X)$. Thus, we only need to show that A is 1-1 and has a closed range. The fact that A is 1-1 is trivial. The closed range is also quite simple: If $(Ax_n)_n$ converges in Y, then it is Cauchy. Since A is unitary $(x_n)_n$ is Cauchy. Since X is Hilbert, $(x_n)_n$ converges, say to x. Since A is continuous, $(Ax_n)_n$ converges to Ax, hence $\operatorname{ran} A$ is closed. □

(b) Give a constructive proof that a unitary map is invertible by showing that $A^*A = 1$. Hint: use the definition of A^* in ips: A^*y in the linear functional in $X^* = X$ that satisfies

$$\langle x, A^*y\rangle := \langle Ax, y\rangle, \quad x \in X.$$

Then prove that $\langle (1 - A^*A)x, x\rangle = 0$, for every $x \in X$. Then try to conclude that $\langle (1 - A^*A)x, x'\rangle = 0$, for every $x.x' \in X$.

Proof: $\langle (1 - A^*A)x, x\rangle = \langle x, x\rangle - \langle Ax, Ax\rangle = 0$, since A is unitary. Now, let x, y be arbitrary, $C := 1 - A^*A$. First, we have $C^* = C$, since $\langle A^*Ax, y\rangle = \langle Ax, Ay\rangle = \langle x, A^*Ay\rangle = \langle (A^*A)^*x, y\rangle$. Then

$$0 = \langle C(x + y), x + y\rangle = \langle Cx, x\rangle + \langle Cy, y\rangle + \langle Cx, y\rangle + \langle Cy, x\rangle.$$

The first two summands equal 0, and since $\langle Cy, x\rangle = \langle y, C^*x\rangle = \langle y, Cx\rangle$, we conclude that $\operatorname{Re}\langle Cx, y\rangle = 0$. Choosing $y := Cx$, shows that $Cx = 0$, hence $C = 0$, hence $1 = A^*A$, as claimed. □

Problem 3. *Let A be a unitary map as in Problem 2.*

(a) Show that AA^* is the orthogonal projector of Y onto $\operatorname{ran} A$. From that, derive a necessary and sufficient condition for AA^* to the identity.

Proof: From Problem 2, $A^*A = 1$, hence $(AA^*)(AA^*) = A(A^*A)A^* = AA^*$. Thus, $C := AA^*$ is a projector. As in Problem 2, $C^* = C$. To prove that C is orthogonal, we need to prove that $\ker C \perp \operatorname{ran} C$. This is true since, if $y \in \ker C$, then

$$\langle Cx, y\rangle = \langle x, Cy\rangle = 0.$$

To prove that $\operatorname{ran} C = \operatorname{ran} A$ it is sufficient to show that $\ker C = \ker A^*$. Obviously, $\ker C \supset \ker A^*$. Now, if $y \in \ker C$, then

$$0 = \langle y, Cy\rangle = \langle A^*y, A^*y\rangle,$$

hence $A^*y = 0$.

Now, $AA^* = 1$ iff the projector in onto the entire space. Since the range of the projector is $\operatorname{ran} A$ the condition we need is $\operatorname{ran} A = Y$. □

(b) Recall that a set $\Phi = (\phi_j)_{j=1}^{\infty}$ of an Hs is a tight frame if the map

$$T^* : X \to \ell_2(\Phi) : x \mapsto (\langle x, \phi \rangle)_{\phi \in \Phi}$$

is unitary. Show that for $x \in X$ the sequence $T^* x$ has minimal norm among all sequences $a \in \ell_2(\Phi)$ that satisfy

$$x = \sum_{\phi \in \Phi} a(\phi)\phi. \tag{1}$$

Hint: Take $A := T^*$ in (a), and compute explicitly A^*. Then use the result in (a). Your proof should show that the sum in (1) converges in X regardless of the order of the elements in Φ.

Proof: With $A := T^*$, we know that A is unitary. Fix $f \in \Phi$ and define $\delta \in \ell_2(\Phi)$ to be the sequence that assumes the value 1 at f and 0 elsewhere. Then, for $x \in X$,

$$\langle A^* \delta, x \rangle = \langle \delta, Ax \rangle = \langle f, x \rangle.$$

This shows that $A^* \delta = f$, hence, by linearity, that

$$A^* a = \sum_{\phi \in \Phi} a(\phi)\phi.$$

Now, if $x = A^* a$ for some $x \in X$ and $a \in \ell_2(\Phi)$, then, by (a), $Ax = (AA^*)a$ is the orthogonal projection of a on $\operatorname{ran} A$, hence $\|Ax\| \le \|a\|$.

Problem 4. *Recall that a compactly supported $w \in L_2(\mathbb{R})$ has m vanishing moments if*

$$\int_{\mathbb{R}} w(t)p(t)\,dt = 0,$$

for every polynomial $p \in \Pi_{m-1}$. Recall that we proved in class that if $\operatorname{supp} w \subset [A, B]$, and if $f \in C^{(m)}([A, B])$, then

$$|\langle f, w \rangle| \le \frac{1}{m!} \|D^m f\|_{L_\infty([A,B])} \|w\| (B - A)^{m+1/2}.$$

Now, assume that $f \in C^{(m)}(\mathbb{R})$, and, in addition, $\operatorname{supp} f \subset [0, 1]$. Assume also that $\operatorname{supp} w = [-A, A]$ for some integer $A > 0$, that $\|w\| = 1$, and that w is bounded (and that, as before, w has m vanishing moments). Recall that, for $j, k \in \mathbb{Z}$, $w_{j,k}$ is defined by

$$w_{j,k} : t \mapsto 2^{j/2} w(2^j t - k).$$

For every fixed $j \in \mathbb{Z}$, provide an estimate on the $\ell_2(\mathbb{Z})$-norm of the sequence

$$k \mapsto \langle f, w_{j,k} \rangle.$$

I.e., estimate $(\sum_{k\in\mathbb{Z}} |\langle f, w_{j,k}\rangle|^2)^{1/2}$. *Hint: consider separately the case of* $j<0$ *and the case* $j \ge 0$.

Proof: For $j \ge 0$, the support of $w_{j,k}$ intersect $[0,1]$ only if $-A < k < 2^j + A$. Also, $\|w_{j,k}\| = \|w\| = 1$. The estimate above implies that, with $C := \frac{\|D^m f\|_{L_\infty(\mathbb{R})}}{m!}(2A)^{2m+1}$,

$$A_j^2 := \sum_{k=-\infty}^{\infty} |\langle f, w_{j,k}\rangle|^2 \le \sum_{k=-A+1}^{2^j+A-1} \frac{\|D^m f\|_{L_\infty(\mathbb{R})}}{m!} \left(\frac{2A}{2^j}\right)^{2m+1} = C\,(2^j + 2A - 1)\,2^{-(2m+1)j}.$$

For $j \ge \log_2(A)$. we can estimate

$$A_j \le 2\sqrt{C}2^{-mj}.$$

So, while the number of non-zero entries grows exponentially with j, the total "energy" A_j decays exponentially with j!

For $j < 0$, there are $2A$ values of k for which $\operatorname{supp} w_{j,k}$ overlaps with $[0,1]$. We can estimate $\langle w_{j,k}, f\rangle$ for each such k by $\|w_{j,k}\|_{L_\infty} \|f\|_{L_1([0,1])}$. Setting

$$K := \|w\|_{L_\infty} \|f\|_{L_1([0,1])},$$

we know that K is finite, since w is bounded. We also know that $\|w_{j,k}\|_{L_\infty} = 2^{j/2}\|w\|$. The final bound in this case is then

$$A_j \le \sqrt{2A}K2^{j/2}.$$

This also goes to zero exponentially fast (as $j \to -\infty$), but not as nearly as fast as the other case. □

Bonus: Let's see that the wavelet representation of f is sparse. For $n > 0$, let's try to estimate the number $N(n)$ of wavelet coefficients $\langle f, w_{j,k}\rangle$ whose modulus is $\ge 2^{-n}$. We expect this number to grow exponentially with n, i.e., to be behave like $\text{const}2^{n\alpha}$. We are interested in finding the parameter α, which can be fetched by computing $\limsup \frac{\log_2(N(n))}{n}$. The limsup above does not depend on A, C and K so, for convenience, we chose $K = C = 1$ and $A = 1/2$.

For $j \ge 0$, $|\langle w_{j,k}, f\rangle| \le 2^{-(m+.5)j}$. Hence, if $j > n/(m+.5)$, all the wavelet coefficients are too small. The total number of wavelet coefficients $\langle w_{j,k}, f\rangle$, $j \ge 0$, that need to be counted is about $2^{n/(m+.5)}$ (why?). For $j < 0$, the wavelet coefficients fall below the threshold once $j < -2n$. The total number here is about $2n$ (should be $2A \times 2n$, but we assume $2A = 1$). This number is negligible compared to the first one. So, $\alpha = 1/(m+.5)$ here. Everything depends on the vanishing moments (and the smoothness of f)! □